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THE MATHEMATICS TEACHER

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THE MATHEMATICS TEACHER

Volume XXXIX Number 4

Edited by William David Reeve

Continuous Transformations of Regular Solids

By H. v. Baravalle
Adelphi College, Garden City, N. Y.

In the development of modern mathematics transformations of different kinds have gained increasing importance and many new concepts have been derived from their studies, for instance, the concept "Invariance of Functional Form." From the point of view of mathematical education, transformations provide stimulating material for experimenting, investigating, and for exercises in visualization. In the following article, transformations of regular solids are carried out which have their bearing on crystallography as they keep the angles of the intersecting planes constant and vary their distances, the characteristics of the variations in crystal forms. Through these transformations, relationships between different kinds of solids will become manifest, for instance, between a tetrahedron and a cube, or between an octahedron and a rhombic dodecahedron. In some of the following transformations no less than five different regular and semi-regular solids evolve from one another in a continuous change. Constructions of descriptive geometry furnish the tools to draw the various solids.

The first transformation starts with a cube. In order to obtain an orthographic projection of a cube which will give a plastic effect, the constructions shown on Plate 1 are needed. They start with a square, the ground plan of the cube (upper left on Plate 1). From the ground plan we derive the elevation of the cube by dropping perpendiculars from all the vertices of the square. Those determine the positions of the vertical edges of the cube. Their length in the elevation equals the length of the sides of the square. From ground plan and elevation we obtain a general orthographic projection (Plate 1, right side) with the help of two scales. One of them determines the left-right position of the vertices of the general projection and it is drawn underneath. This scale has been transferred from the ground plan where it has been formed by the distances between horizontal lines drawn through the vertices of the square. The other scale, drawn on the right side of the general projection, determines the relative altitude of the vertices. It has been transferred from the elevation and been obtained from the distances between parallel lines drawn through the vertices of the elevation under any chosen angle. The assembling of the two scales by which the vertices of the general projection are obtained is carried out with the help of numbers one to eight, attached to the vertices of a cube. The numbers 1–4 refer to the base square, the numbers 4–8 to the top square of the cube. In the ground plan the base and the top

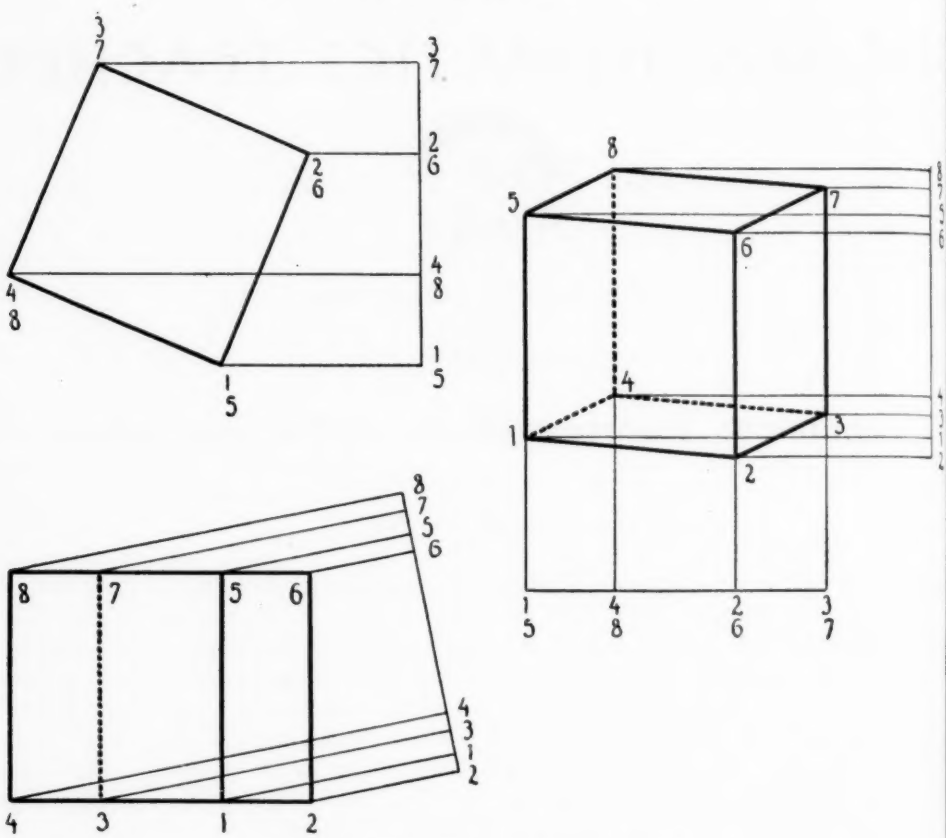

PLATE 1. Orthographic projection of a cube.

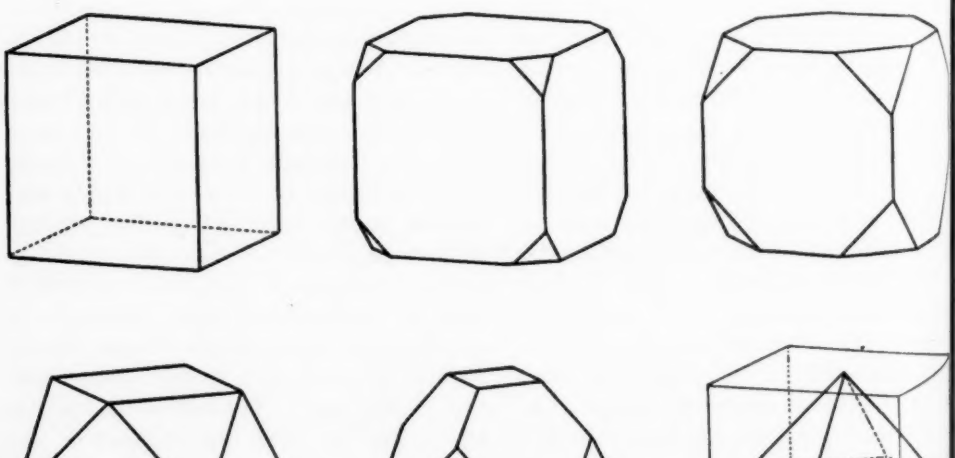

PLATE 2. Cutting the corners of the cube.

squares coincide and the vertices carry two numbers each: 1 & 5; 2 & 6; 3 & 7; 4 & 8. In the elevation the same pairs of points include the vertical edges. All the numbers are carried over to the scales and are transferred with them to the general projection. Where two auxiliary lines drawn from the points with the same numbers on both scales intersect, we obtain a vertex of the general projection.

The general projection of a cube is carried over to Plate 2 (upper row left). Now we start a transformation by cutting off all the corners of a cube, so that all sections form equilateral triangles. The line segments which are cut off from the three edges which meet in each corner are all equal. The operation is carried out in the orthographic projection through proportional segments. If we cut off $\frac{1}{6}$ of each edge at each corner of the cube, we join the points which have the distances of $\frac{1}{6}$ of the lengths of the edges from the vertex. Thus we obtain the second form on Plate 2 (upper row middle). The newly formed edges around the triangles are at that point still relatively small compared with the remaining parts of the cube edges, but the deeper we cut into the cube the larger their size becomes and the smaller are the remaining cube edges. The further pursuit of this change necessarily leads to a state in which the edges around the triangles become equal to the remaining sections of the cube edges. Then the octagons formed at the faces of the cube become regular and the result is a semi-regular solid with six octagonal and eight triangular faces. It is shown as the third form on Plate 2 (upper row right). The ratio between the segments of the edges of the cube which have been cut off and the original edges is now $1-\frac{1}{\sqrt{2}}$. This follows from the construction of a regular octagon inscribed into a square. On Plate 3 (left side) one of the faces of the cube is drawn, first without projective distortion. To the square the diagonals are added and half the diagonals are transferred through arcs (centers in the vertices of the square) to the sides of the square. The points so obtained are the vertices of the regular octagon. The side of the square being "a" each diagonal is $\sqrt{2}\cdot a$ and the radii of the arcs are $\frac{\sqrt{2}\cdot a}{2}=\frac{a}{\sqrt{2}}$, the segments cut off at each vertex from each side of the

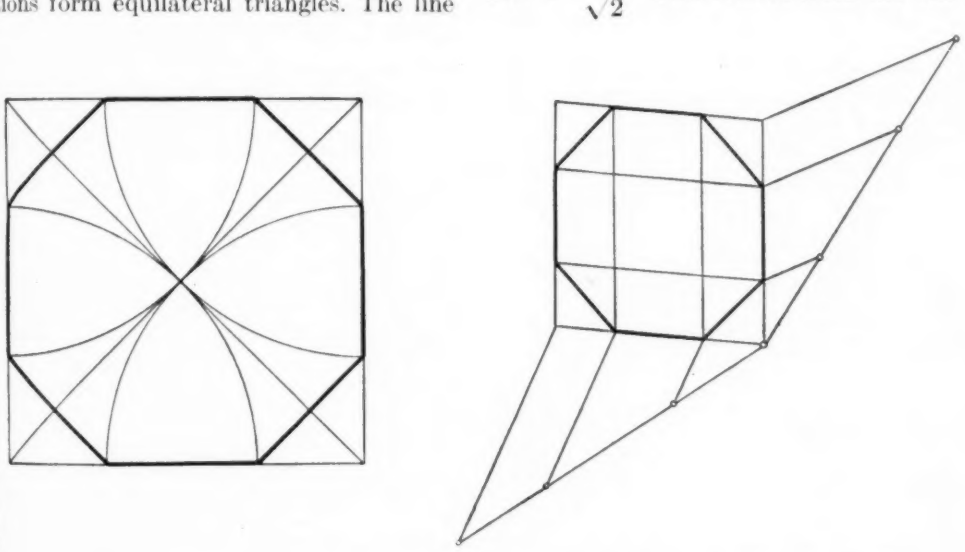

PLATE 3. Inscribing a regular octagon into a square in the orthographic projection.

square are $a-\frac{a}{\sqrt{2}}=a\left(1-\frac{1}{\sqrt{2}}\right)$, and the remaining parts of the sides of the square:

$$a-2\cdot a\left(1-\frac{1}{\sqrt{2}}\right)=a\left(1-2+\frac{2}{\sqrt{2}}\right)$$
$$=a(\sqrt{2}-1).$$

The other sides of the octagon can be figured out as the diagonals of the little square of which one-half is cut off at each corner: $\sqrt{2}\cdot a\left(1-\frac{1}{\sqrt{2}}\right)$. Multiplying $\sqrt{2}\cdot\left(1-\frac{1}{\sqrt{2}}\right)$ we obtain the same expression as before: $a(\sqrt{2}-1)$. The right side of Plate 3 shows the orthographic projection of a face of the cube to which the ratios of the regular octagon have to be transferred. Through one of its vertices two straight lines are drawn under any chosen angle and on those the line segments obtained from a side of the square on the left of Plate 2 are transferred. The end of the last segment is joined with the end the adjacent side of the face and the subdivisions are obtained through parallel lines, on the principle of similar triangles. Thus the projection of the semi-regular solid in the third diagram of Plate 2 has been constructed.

Continuing further with the transformation, by cutting deeper into the cube, the triangular faces increase, whereas the remaining parts of the cube edges decrease until they finally disappear. This is the case when the vertices of the triangles moving along the cube edges reach their middle points. The form so obtained is a semi-regular solid with six triangular and six square faces. It is a cube octahedron shown as the fourth diagram on Plate 2 (lower row left).

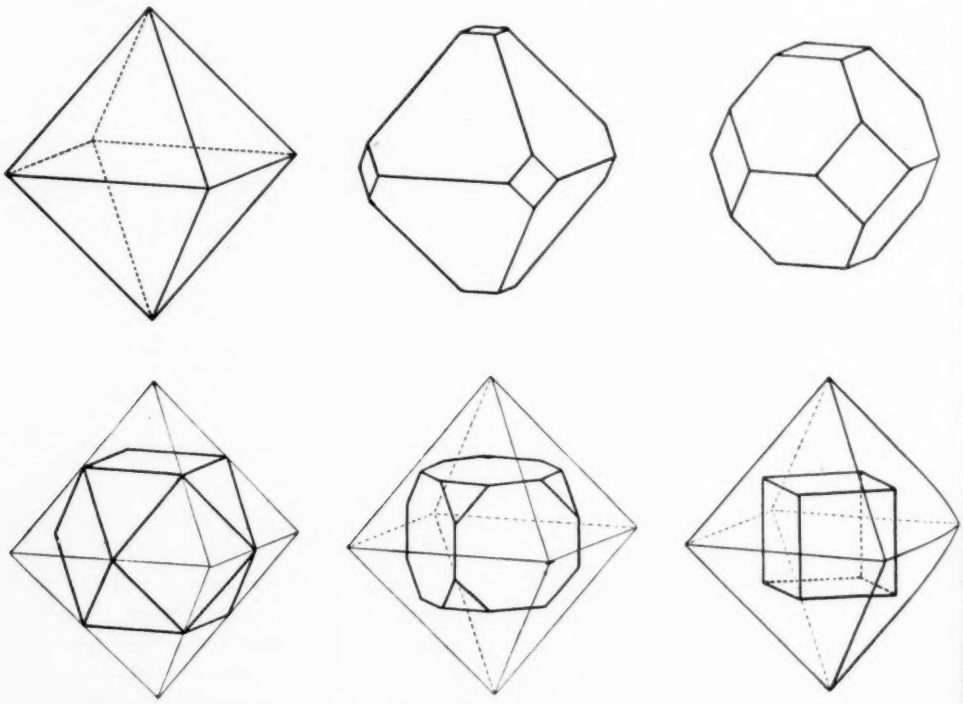

PLATE 4. Cutting the corners of an octahedron.

Carrying on the transformation beyond this point the triangles turn into hexagons, as new edges are formed between them. First the new edges are smaller than the

others but they gradually increase whereas the others decrease. When all the sides have become equal, the fifth form on Plate 2 is obtained. It is a semi-regular solid, formed by eight hexagons and six squares. The ratio between the way covered by each point of intersection along the cube edges and the total lengths of the edges is now three-quarters. Continuing the transformation through its last quarter, the size of the squares which are left over from the faces of the cube diminishes further, whereas the edges between the hexagons increase. Finally the squares disappear and the hexagons become triangles. The solid then obtained is an octahedron. It is shown, together with the original cube as the sixth form on Plate 2 (lower row right). With it the transformation comes to an end as a further continuation of the motion of the intersecting planes merely affects the size of the octahedron.

The same kind of transformation carried out with an octahedron is shown on Plate 4. The corners of the octahedron are cut off and the intersecting planes, forming square sections at all the corners, gradually cut deeper into the cotahedron. The transformation is shown in six phases and ends with a cube.

On Plate 5, the process is once more repeated with a tetrahedron (upper row left). Its four corners are cut off and four new triangular faces are added. The triangular faces of the original tetrahedron become hexagons (upper row middle). The edges around the new triangles increase, those between hexagons decrease. Thus, the solid approaches the state in which all edges become equal. It is reached when the vertices traveling along the edges of the tetrahedron have covered one-third of their size. On Plate 5, this is shown as the third form (upper row right). It is a semi-regular solid with four triangular and four hexagonal faces. Continuing still further with the motion, the triangles keep increasing, the hexagons diminishing, and when each vertex has covered one-half of

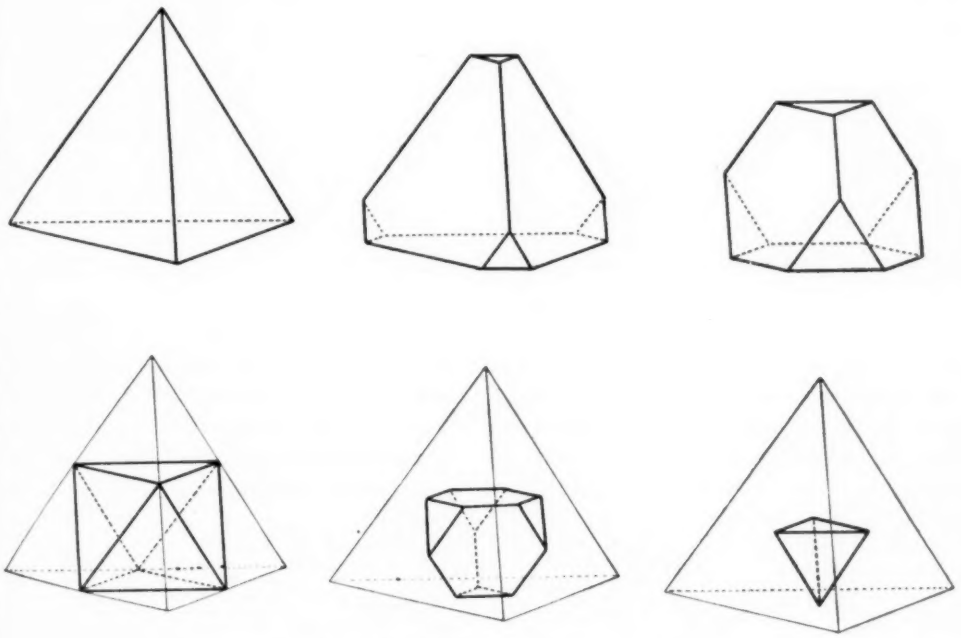

PLATE 5. Cutting the corners of a tetrahedron.

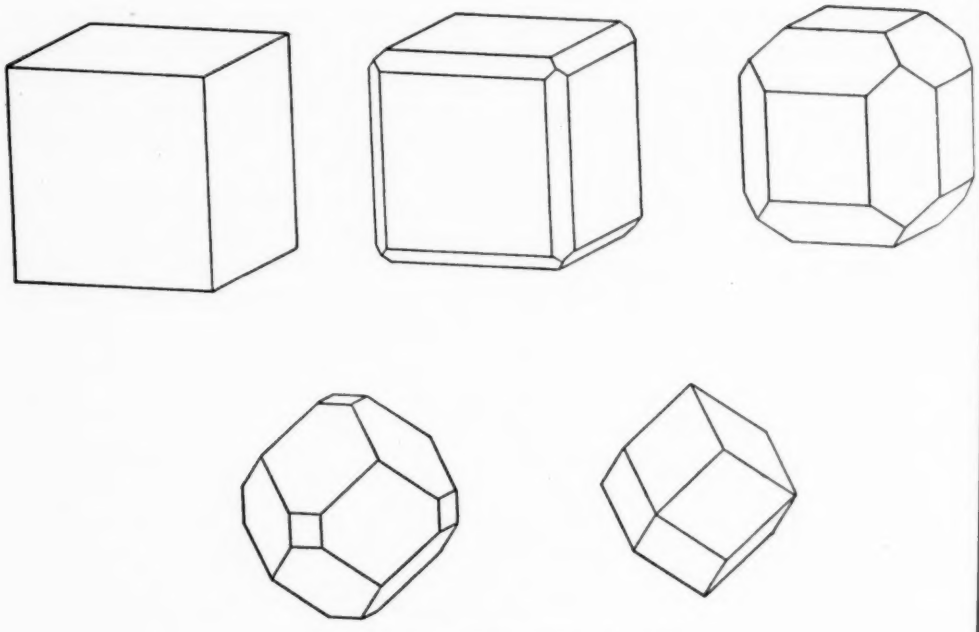

PLATE 6. Cutting the edges of a cube.

a tetrahedron edge, two vertices meet in the middle of each edge. Then the hexagons become triangles and the solid turns into an octahedron (lower row left on Plate 5). The octahedron occupies a similar position in regard to the tetrahedron as the cube-octahedron to both the cube and the octahedron. In its further transformation the solid once more becomes semi-regular with four hexagonal

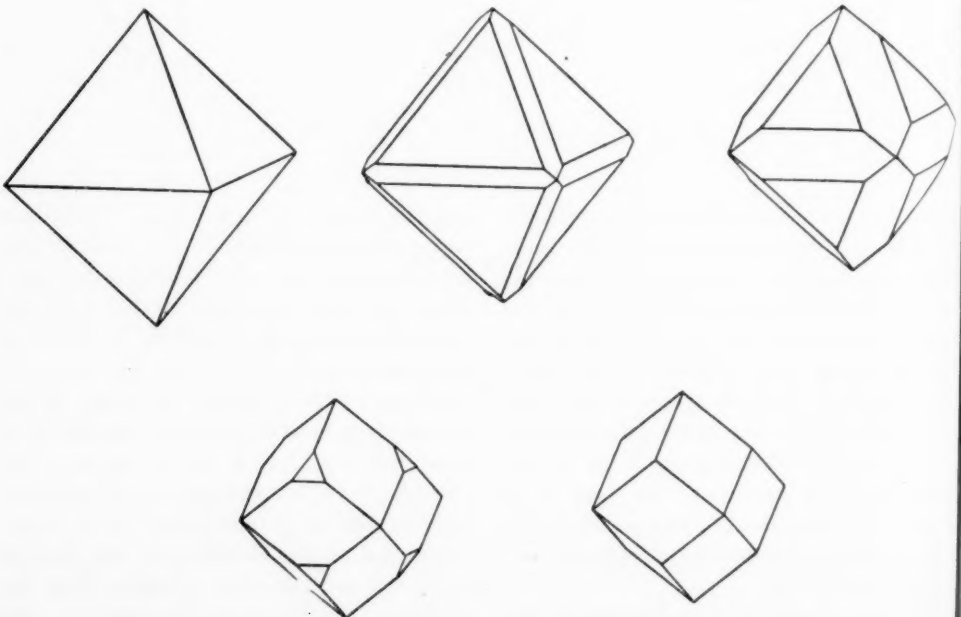

PLATE 7. Cutting the edges of an octahedron.

and four triangular faces (lower row middle) and finally, turns back into a tetrahedron (lower row right).

Other kinds of transformations with solids can be carried out by—instead of cutting the corners—cutting the edges. This is done by moving intersecting planes simultaneously at all the edges into the solid (each intersecting plane being parallel to an edge and perpendicular to the plane bisecting the angle between the planes of the adjacent faces). These successive phases of the transformation are shown on Plate 6. Starting with a cube (upper row left) we obtain, by cutting its edges, a solid with six square and twelve hexagonal faces (upper row middle). By cutting deeper into the cube, the square faces diminish and the hexagons are widened. At the state shown as the third form on Plate 6 (upper row right) the edges of the squares are still longer than those formed between the hexagons. Later on, the relative sizes are reversed (lower row left). The squares continue to diminish until they finally disappear. Then the hexagons become rhombi. The solid gets twelve rhombic faces and becomes the rhombic dodecahedron which plays an important role in crystallography.

Applying the transformation of cutting the edges, to an octahedron (Plate 7, upper row left) we first obtain a solid with eight triangular faces and twelve hexagonal ones, shown as the second form on Plate 7 (upper row middle). The triangles diminish and the hexagons widen. At the state shown as the third one on Plate 7 (upper row right) the edges around the triangles are still longer than those between the hexagons. Later on their mutual sizes are reversed. This is the case at the fourth solid of Plate 7 (lower row left). Finally the triangles disappear and the resulting solid is again a rhombic dodecahedron (lower row right).

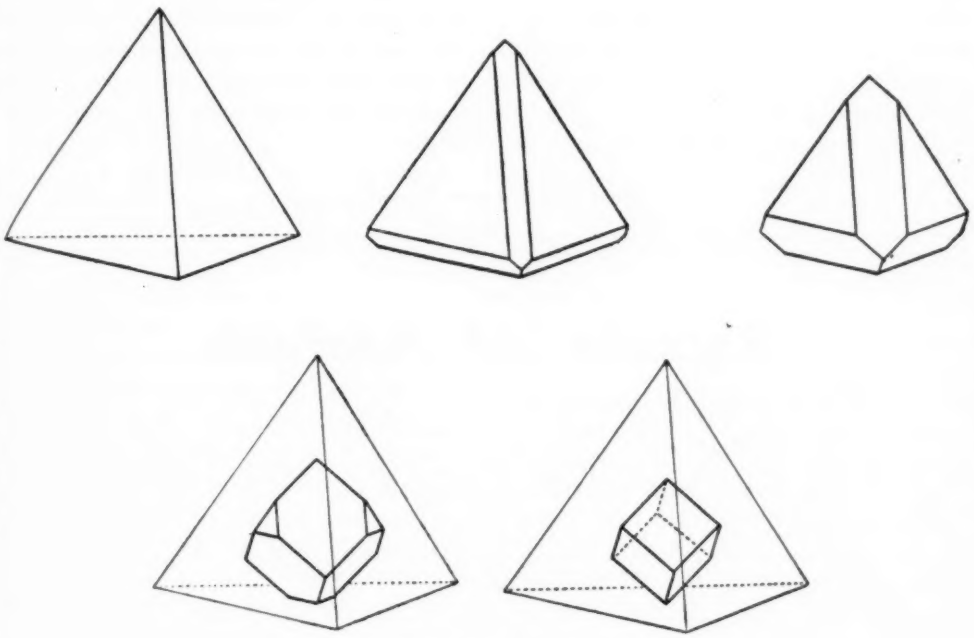

PLATE 8. Cutting the edges of a tetrahedron.

Cutting the edges of a tetrahedron (Plate 8, upper row left) we obtain solids with four triangular and six hexagonal faces. First the hexagons have narrow elongated forms as in the second diagram on Plate 8 (upper row middle). Later the

hexagons gradually widen. The solid passes through a third state (Plate 8 upper row right), and a fourth state (lower row left) and finally turns into a cube (lower row right). Thus a relationship between the tetrahedron, the prototype of a solid formed by triangles, and the cube, the prototype of a solid formed by squares. is established. This relationship is anticipated in the fact that the tetrahedron has six edges and the cube six faces and that the sum of vertices plus faces in the tetrahedron is eight, which is the same as the amount of vertices in the cube. At every edge of the tetrahedron originates a face of the cube, whereas a vertex of the cube is formed for each vertex and at each face of the tetrahedron.

The connection between cube and octahedron on one hand, and the rhombic dodecahedron on the other is also based on number relations. Both the cube and the octahedron have twelve edges. The rhombic dodecahedron has twelve faces. For both the cube and the octahedron the sum of the numbers of vertices plus faces is fourteen, and the rhombic dodecahedron has fourteen vertices. The mutual relationship between the cube and the octahedron of the first transformation (Plates 3 and 4) is anticipated in the fact that the cube has eight vertices and the octahedron, eight faces, and that the cube has six faces and the octahedron, six vertices. In the case of the tetrahedron, the number of both faces and vertices is four which leads to the reciprocal relationship with itself.

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Mathematical Probabilities in Games of Chance
The Game of Sevens

By E. Bonavent Dagobert
San Francisco, Calif.

In a letter to Pascal from a courtier, friend of Mme. de Maintenon and gambler, we read the following: "Remember that I am discovering things which you would have never seen for yourself, things so rare that the wisest ancients completely ignored them and at which the greatest modern mathematicians have been surprised." The writer was Antoine Gombault, Chevalier de Méré (1610–1684). The "things so rare" were the gambling chances in a dice game. The things the wisest ancients completely ignored were the problems presented by random events, and the things at which the greatest modern mathematicians have been surprised were the heretofore unexplored realms of chance, of mathematical probabilities, which the keen intellects of Pascal, Fermat, De Moivre, Bernoulli, Lagrange and Laplace organized into an important science, the science of Modern Statistics.

But returning to the gambler and to his game of dice, the "Game of Sevens" as it was then called, and which precipitated such signal events, this is what he found and what anyone can find for himself when casting two dice in a long series of trials: the sum seven will appear more often than any other sum, or rather seven and its neighboring values six and eight, while two, three, eleven and twelve will be far rarer events. However when casting one die, the numbers will all be equally frequent. Why is this so? Should we not, the dice being what they are, always expect equal frequencies for a number thrown at random? Is there a hidden law which attracts seven more often than two or three? This, precisely was the question the chevalier de Méré asked of his friend Pascal, and out of this question the theory of Mathematical Probabilities was born, a theory capable of making accurate predictions regarding frequencies of random events. In the case of casting sums with two dice, the theory foresees the behavior of seven as being "normal." It singles it out as the most frequent and hence as the most probable of all the other sums before the dice are cast, together with its immediate neighbors eight and six.

But there are other features in the Game of Sevens which can be brought to light by the use of mathematical probabilities, surprising features from the standpoint of chance and analysis as well, since the game is widely played, yet rarely studied. It is none other than the lowly "Shooting Craps" which is based on the central idea of casting sevens either as winners or losers. The first discovery which will be made when computing the probabilities for winning or losing will be that they are equitable to an amazing degree. In other words the odds for winning are nearly equal to the odds for losing. In spite the great variety of winning sums at "Craps" we might as well toss a penny for "heads or tails" as far as the gambling chances are concerned. This, of course, is by no means obvious, as the following discussion will reveal.

The basic game is played by two partners, with two dice, according to the following rules: If the two partners are identified as A, the player, and B, the banker, then A wins when casting one of the "Naturals" seven or eleven. He loses when casting one of the "Craps," two (snake eyes), three (trey), or twelve (boxcars). After a loss the dice change hands. When A does not cast a Natural, he may still win by casting four (Little Joe), five

(Phoebe), six, eight, nine or ten. In this event namely he continues playing until he repeats his first sum. If a seven turns up in the series of trials, he loses and the dice change hands. While no seven shows, he continues trying to duplicate his first throw. What are the probabilities of the two players, their gambling chances?

Webster's New International Dictionary (Second Edition Unabridged, page 619) quotes the odds in favor of winning "at Craps" to be 244/495 as against 251/495 in favor of losing. How, we may ask, is this estimate arrived at and is it correct? The following investigation will answer this question. Moreover it will specify numerically each possible event on the winning or losing side. It will show that all the probabilities tend towards finite limits in an infinite number of trials, limits, it will be seen, of convergent geometric progressions. And again, the gambling chances not only continue equitable, in prolonged series of trials, but actually become more so as the game progresses and the dice change hands.

The tabulation below contains all the possible combinations of points cast with two dice and gives moreover the probabilities for each sum at a glance.

Sums cast with two dice

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

The first horizontal row indicates the points on one die while the first vertical column the points on the other. Entering the table with these two arguments, we obtain the sums inside the field. By following the arrows, we see the probability for any sum i, called P(i) immediately to be:

$P(2) = 1/36$ $\quad P(8) = P(6)$
$P(3) = 2/36$ $\quad P(9) = P(5)$
$P(4) = 3/36$ $\quad P(10) = P(4)$
$P(5) = 4/36$ $\quad P(11) = P(3)$
$P(6) = 5/36$ $\quad P(12) = P(2)$
$P(7) = 6/36$

The probabilities for the Naturals seven or eleven are: $(6+2)/36 = 2/9$. The probabilities for the Craps, two, three, twelve, are: $(1+2+1)/36 = 1/9$. The probabilities for the Repeat numbers, R, namely four, five, six, eight, nine, or ten are: $(3+4+5+5+4+3)/36 = 2/3$. The probabilities for winning with these R's require further study. They depend on the number of trials necessary to repeat the given R. Two throws may suffice.

Clearly then, the probabilities for winning in two throws are:

on Four: $P(4)P(4) = (3/36)\ (3/36)$
on Five: $P(5)P(5) = (4/36)\ (4/36)$
on Six: $P(6)P(6) = (5/36)\ (5/36)$
on Eight: as on Six
on Nine: as on Five
on Ten: as on Four.

If more than two trials are necessary, the pattern for winning is R-R, R-R-R---R, and so forth. The probabilities for the R's are known. Those for the blanks are the probabilities for the neutral events, "neither winning nor losing," which keep the series going. They are easily determined for each of the six R's by the condition: neither R, nor seven. Identified as N(i) they are:

$N\ (4) = (36\text{-}6\text{-}3)/36 = 27/36 = 3/4$
$N\ (5) = (36\text{-}6\text{-}4)/36 = 26/36 = 13/18$
$N\ (6) = (36\text{-}6\text{-}5)/36 = 25/36$
$N\ (8) = N(6)$
$N\ (9) = N(5)$
$N(10) = N(4)$

The probabilities for winning in one, two, three, or more trials appear therefore in the general form: $PP+PNP+PNNP+PNNNP+\cdots$, or $PP(1+N+N^2+\cdots)$.

Substituting the numerical values for P and N, we obtain the following geometric progressions:

$(3/36)(3/36)(1+3/4+(3/4)^2+\cdots)$
$(4/36)(4/36)(1+13/18+(13/18)^2+\cdots)$
$(5/36)(5/36)(1+25/36+(25/36)^2+\cdots)$

The sums of these series are 4, 18/5, 36/11 respectively.

The probabilities for winning the series of Eight, Nine, Ten, are as those of Six, Five, Four, respectively. The total for winning on any R in any number of trials is then: $2(3/36)^2(4)+2(4/36)^2 18/5+2(5/36)^2(36/11)=1/18+4/45+25/198=(110+176+250)/1980=536/1980$.

Since the probability for winning on Naturals was previously found to be 2/9, the total probability for winning the game on Naturals or Repeats, in any number of trials is $536/1980+2/9=(536+440)/1980=976/1980=244/495$.

This is the value quoted above in Webster's. But it is the mathematical probability for the event of winning the game in any possible manner, and not "the odds" as stated in Webster. The odds for winning are 244/251. This anticipates the result of the following paragraph where the mathematical probability for losing the game will be developed, the odds being the quotient of the two fractions.

A simple calculation along similar lines will quickly yield the chances for losing. Utilizing the results obtained in the previous discussion, we first give the general form for the probability of breaking the series. This form will be: $PP+PNP+PNNP+\cdots=PP(1+N+N^2+\cdots)$.

Substituting the numerical values for the first P, $P=P(i)$ and for the second P, $P=P(7)=1/6$, and referring to the geometric series by S(4), S(5) and S(6), we obtain the chances for losing, in the concise form:

on Four and Ten: $2(3/36)(1/6)(S_4)$
on Five and Nine: $2(4/36)(1/6)(S_5)$
on Six and Eight: $2(5/36)(1/6)(S_6)$

to which must be added the probability for losing on Craps, namely 1/9. Thus the total probability for losing the game in any possible manner is found in the form: $2(3/36)(1/6)4+2(4/36)(1/6)(18/5)+2(5/36)(1/6)(36/11)+1/9=1/9+2/15+5/33+1/9=(55+66+75+55)/495=251/405$.

A valuable check on the computations is furnished by the knowledge that the probabilities for winning and losing must add to one. Indeed, we have $244/495+251/495=495/495=1$. This was anticipated when stating the odds, but it is now established numerically.

If these final probabilities are expressed in decimal notation, the amazing properties of this game of chance are seen in a truly startling manner, since the fractions become 0.49 and 0.51, so that the odds in favor of winning are 49 to 51. The odds for losing are 51 to 49, so that throughout the game the player "works" for the banker. Yet at the moment of losing, the player becomes the banker, and the chances are reversed. Hence, the game is not only equitable to an unusual degree but becomes more so as it proceeds. Only when the game is played against a "board," that is against a permanent banker does it cease to be equitable, the winning chances remaining constantly in favor of one side. In that case we have the chances of a lottery, where "the banker always wins."

The chances for winning on specified series are extremely small. Moreover they are nearly equal. Thus, the chances for winning on "Little Joe" are three in a hundred, on "Phoebe," four in a hundred, on Six, six in a hundred. Yet, combined, they are again neatly balanced, as will be seen from the following tabulation.

Here then, is the mechanism of this remarkable game truly revealed. Its pulse is the casting of sevens. Seven is the pivot around which the chances are balanced, making and breaking the game frequently by its superior probability. If this is obvious, not so, indeed is the fact that:

(1) the frequency of seven as loser is more than twice its frequency as winner.

(2) that winning on Naturals is nearly

Events	Probabilities of Events	Total Probabilities	
Winning on:			
4 . . . 4	0.03		
5 . . . 5	0.04		
6 . . . 6	0.06		
8 . . . 8	0.06		
9 . . . 9	0.04		
10 . . . 10	0.03		
Winning on Series	*0.26*		
Winning on:			
7	0.17		
11	0.06		
Winning on Naturals	0.23		
Winning on either Naturals or Series		*0.49*	
Losing on:			
4 . . . 7	0.06		
5 . . . 7	0.07		
6 . . . 7	0.07		
8 . . . 7	0.07		
9 . . . 7	0.07		
10 . . . 7	0.06		
Losing on Series	*0.40*		
Losing on:			
2	0.03		
3	0.05		
12	0.03		
Losing on Craps	*0.11*		
Losing on either Craps or Series		*0.51*	
Winning Game or Losing Game			*1.00*

as probable as winning on Series, no matter how many trials are made.

(3) that the chances for specified series are small and nearly equal.

(4) and that the partner who waits for his turn is more apt to win than the player himself.

These points, together with their numerical values are the result of an investigation which only past centuries of research into the nature of random events have made possible.

New Contributions in Mathematics to Encyclopaedia Britannica

SIX EXPERTS on subjects related to mathematics have made their first contributions to the Encyclopaedia Britannica for the 1946 printing of the 178-year-old reference work.

The new Britannica contributors and their topics are C. C. MacDuffee, professor of Mathematics at the University of Wisconsin: "complex numbers"; Ralph G. Sanger, assistant professor of Mathematics and dean of students in the Division of Physical Sciences: "ellipse," "parabola," "paraboloid," and "symmetric functions"; O. F. G. Schilling, assistant professor in Mathematics at the University of Chicago: "elliptic functions"; Ivan Stephen Sokolnikoff, professor of Mathematics at the University of Wisconsin and technical aide for the Applied Mathematics Panel of the National Defense Research Committee: "vector analysis"; Lawrence Murray Graves, professor of Mathematics at the University of Chicago: "Taylor's theorem"; and Dirk Brouwer, editor of the "Astronomical Journal," Munson professor of Natural Philosophy and Astronomy and director of the Observatory at Yale University: "celestial mechanics."

Several thousand experts have revised, rewritten or prepared new articles for the 1946 printing of the Encyclopaedia Britannica under its plan for continuous revision.

The First Calculus Textbooks

By Carl B. Boyer
Brooklyn College, Brooklyn, N. Y.

The year 1946 marks just a quarter of a millennium since the appearance of the *Analyse des infiniment petits* of L'Hospital.[1] This book, the first published text on the calculus, could boast truly that in its day it filled with distinction that ubiquitous lure of textbook writers—the long-felt need. Moreover, its influence and popularity dominated the whole of the eighteenth century, the period during which the new analysis developed until it completely overshadowed other branches of mathematics.

Historically integration antedates differentiation by some two thousand years. The ancient Greek method of exhaustion and the infinitesimal mensurations of Archimedes represent early examples of limits of integral sums; but it was not until the seventeenth century that Fermat found tangents and critical points through methods equivalent to evaluating limits of difference quotients. It was the discovery of the inverse nature of these two processes, together with the consequent exploitation of the anti-derivative in determining limits of sums, which leads one to call Newton and Leibniz the inventors of the calculus. Differentiation, both direct and inverse, became the basic algorithm in a new and powerful part of mathematics. Integration was nothing but "the memory of differentiation," and it was not until a century and a half later that some attention again was directed to the summation concept in the calculus.

The earliest expositions of the new analysis were but the barest outlines of the rules of differentiation. The calculus first appeared in print in a six-page memoir by Leibniz in the *Acta Eruditorum* of 1684. This contained a definition of the differential and gave brief rules for determining the differentials of sums, products, quotients, power, and roots. It included also a few applications to problems of tangents and critical points. This account was so short and contained so many misprints as to be largely enigmatic to the best mathematicians of the age.[2] The basis of Newton's method of fluxions was first published unobtrusively in the form of lemmas in the *Principia* of 1687, more than a score of years after his discovery of the calculus. Here one finds some properties of limits, as well as cryptically brief directions for finding infinitely small "moments" of products, powers, and roots.[3]

The early years of the calculus resemble the infancy of analytic geometry in that little was done to popularize the subject. It is said that when Huygens about 1690 wished to master the new methods, there were not half a dozen men qualified to expound this analysis. Newton between 1669 and 1676 had composed several treatises on the elements of fluxions, but these did not begin to appear in print until 1704. John Craig in 1685 and 1693 published two works based in part on the method of Leibniz, but these were not intended as introductions to the subject and were moreover difficult to read because of the geometrical form in which they were cast. On the Continent the Bernoulli brothers with some effort achieved

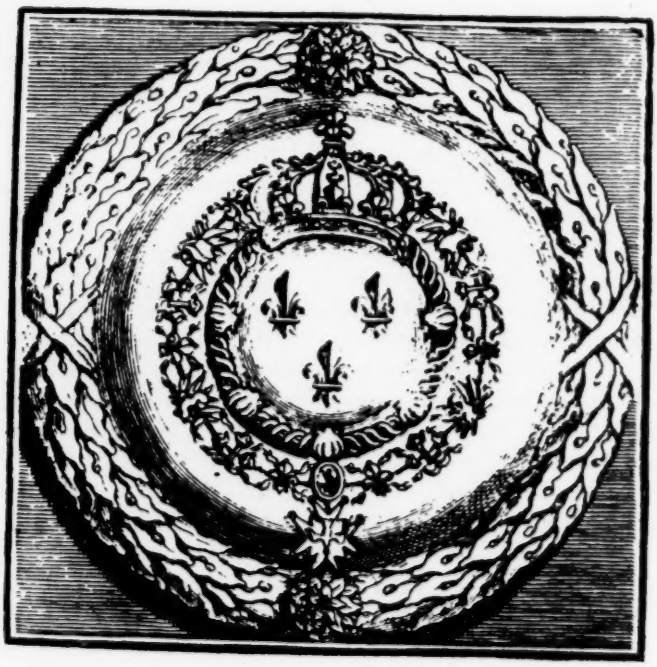
[1] G. F. A. de L'Hospital, *Analyse des infiniment petits, pour l'intelligence des lignes courbes*, Paris, 1696. Page references indicated in this paper refer to the original edition.

[2] See Gustav Eneström, "Über die erste Aufnahme der Leibnizschen Differentialrechnung," *Bibliotheca Mathematica* (3), IX (1908–1909), 309–320. An English translation of Leibniz's paper of 1684 is found in D. E. Smith, *A Source Book in Mathematics* (New York, 1929), pp. 619–626.

[3] Sir Isaac Newton, *Opera quae exstant omnia* (ed. by Samuel Horsley, 5 vols., Londini, 1779–1785), I, 251, II, 277–280.

ANALYSE
DES
INFINIMENT PETITS,

Pour l'intelligence des lignes courbes.



A PARIS,
DE L'IMPRIMERIE ROYALE.

M. DC. XCVI.

TITLE-PAGE OF THE FIRST EDITION OF L'HOSPITAL'S TEXTBOOK
(Courtesy of Columbia University Library)

MARQUIS DE L'HOSPITAL
(Courtesy of David Eugene Smith Library)

a thorough understanding of the subject, and it was Jean Bernoulli who then instructed L'Hospital, while the latter in turn passed his knowledge on to Huygens. That there was an atmosphere of enthusiasm surrounding the calculus toward the close of the seventeenth century is to be ascribed in large part to the research articles published in the learned periodicals of their day by Leibniz, L'Hospital and the Bernoullis. The large mathematical public had, before 1696, no easy introduction to the subject. Jean Bernoulli about 1691–1692 had composed two little textbooks, but their publication was long delayed: that on the integral calculus finally appeared in 1742,[4] and that on the differential calculus almost two hundred years later still.[5] That the text of L'Hospital achieved such prompt success is ascribable largely to the fact that it was the first book to supply what the many had long awaited—an elementary introduction to a new and fascinating branch of mathematics.

Guillaume-François-Antoine de L'Hospital, Marquis de St. Mesme,[6] was born in 1661. His interest in geometry was aroused at an early age, and by the time he was fifteen years old he solved difficult problems on the cycloid which Pascal had proposed. He became a captain of cavalry, but later gave up army life to devote more time to the study of mathematics. When Jean Bernoulli was in Paris in 1692, L'Hospital for four months studied the new infinitesimal geometry under him. From that time on he joined the ranks of the mathematical elite and became the chief exponent of the calculus in France. Recognizing the lack of elementary expository works, he wrote to Bernoulli in 1695 that he was on the point of publishing a work on conic sections and that he proposed to add a little treatise on the differential calculus. L'Hospital indicated that he would render his master the justice he deserved, and he added modestly that the projected work would serve merely as an introduction to a more elaborate treatise, *De scientia infiniti*, which Leibniz planned to write.[7]

The promised book on conics was delayed by the author's illness, and appeared posthumously in 1707; but the *Analyse des infiniment petits* was published promptly in 1696. In the preface L'Hospital points out that he owed much to Leibniz and the Bernoullis, especially "the young professor at Groningen"; and he says that they can take whatever credit they wish, leaving to him what they will.[8] After Jean Bernoulli had received a copy of the book, he wrote and thanked l'Hospital for mentioning him in the volume, and he promised to return the compliment when he in turn should have composed something. He wrote that the work was admirably done, and praised it for the sound arrangement of propositions and for the intelligibility of the exposition. To L'Hospital's suggestion that he should write an integral calculus as a sequel to the *Analyse*, in view of the protracted delay on the part of Leibniz, Bernoulli replied that he was so distraught by domestic embarrassments that he could not carry out the project.[9]

Seldom has a textbook in mathematics been received by savants with such eagerness as was that of L'Hospital. It opened

[4] Jean Bernoulli, *Opera omnia* (4 vols., Lausanne and Geneva, 1742), vol. III; also *Die erste Integralrechnung*. Eine Auswahl aus Johann Bernoullis mathematischen Vorlesungen über die Methode der Integrale und anderes aufgeschrieben zum Gebrauch des Herrn Marquis de l'Hospital in den Jahren 1691 und 1692. Translated from the Latin, Leipzig and Berlin, 1914.

[5] Jean Bernoulli, *Die Differentialrechnung aus dem Jahre 1691/92*, Ostwald's klassiker, No. 211, Leipzig, 1924.

[6] A satisfactory biographical account is found in J. F. Michaud, *Biographie Universelle* (2nd ed., Paris, 1880), XXIII, 448–451. In later years the family name came to be spelled L'Hôpital instead of L'Hospital.

[7] See L'Hospital, *op. cit.*, preface.

[8] *Ibid.*

[9] Eneström, "Sur le part de Jean Bernoulli dans la publication de l'Analyse des infiniment petits," *Bibliotheca Mathematica* (new series), VIII (1894), 65–72. Cf. O. J. Rebel, *Der Briefwechsel zwischen Johann (I). Bernoulli und dem Marquis de l'Hospital*, Bottrop i.w., 1934.

the door to a powerful tool which previously had been accessible only to the very gifted, and thus made it possible for the average reader to solve problems which had defied the best efforts of the geometers of Greece. But the reception was not one of unmixed acclaim. Echoing the contemporary literary controversy on "ancients vs. moderns," some admirers of the classic synthetic geometry cast doubt on the soundness of the new analytic approach. The Académie des Sciences in 1701 was sharply divided over the *Infiniment petits*. The Abbé Gallois led a polemic attack on the methods of the book, ably supported by Rolle.[10] L'Hospital, although he was a member of the Académie, did not reply; but his cause was warmly and effectively sustained by Varignon, Saurin, and others. The controversy reached such a pitch that the Académie finally felt obliged to name a commission to judge the question. Interest in the attack now began to wane, but L'Hospital did not witness the ultimate triumph of the infinitely small, for he died in 1704.

Jean Bernoulli meanwhile had observed with apparent jealousy the growing success of his protégé's book. During the years immediately following the author's death, he wrote letters attacking the work and practically accusing L'Hospital of plagiarism. L'Hospital's general prefatory acknowledgment, Bernoulli argued, did not do him justice inasmuch as he was in effect the true author of almost all the substance of the book. Historians of mathematics have been divided in judgment on this matter, but a thorough analysis of the question made by Enestrom reaches the conclusion that the broad claims of Bernoulli with respect to the authorship of the material are not substantiated.[11]

[10] Académie des Sciences, *Histoire et mémoires*, 1701 (Amsterdam, 1735), Histoire, pp. 114 ff. This controversy was the second of three important attacks on the calculus. The first was that of Nieuwentijdt against the work of Leibniz in 1695–1696. The third was the famous *Analyst* controversy of 1734–1735 concerning the calculus of Newton.

[11] Enestrom, *op. cit.*

The *Analyse des infiniment petits* represents the first systematic treatment of the calculus, and as such it presents a good picture of the level of the subject at that time. The preface includes a brief historical account in which the author, while admitting that "Newton also had a kind of calculus," prefers and emphasizes the contributions of Leibniz. The treatise itself is divided into ten chapters or sections, of which the first (pp. 1–10) is devoted to the fundamental definitions, assumptions, and rules of procedure. A variable quantity is defined loosely as one which continually increases or diminishes; and the infinitely small portion by which a variable quantity changes is called the "difference" or differential. Two postulates are laid down: that one can take as equal two quantities which differ only by an infinitely small quantity; and that a curved line can be considered as an infinite assemblage of infinitely small straight lines which determine, by the angles which they make with each other, the curvature of the curved line. To L'Hospital these premises appeared to be "so self-evident as not to leave the least scruple about their truth and certainty on the mind of an attentive reader," but he adds that if space permitted he could prove them in the manner of the ancients. The crudeness of these fundamental principles is an indication of the infancy of the subject rather than of carelessness on the part of the author. The language of L'Hospital is eminently restrained in comparison with that of contemporaries who wrote such things as

$$\frac{d^3y}{d^2x} = d^3y \int^2 x \quad \text{or} \quad \infty \cdot \infty^{\infty-1} = \infty^{\infty}.$$

The basic differential formulas for algebraic functions—sums, products, quotients, powers, and roots—are derived by L'Hospital in the customary Leibnizian manner, infinitesimals of higher order being neglected. Independent and dependent variables were not always clearly distinguished at the time, and here L'Hospital took a step in advance by stating

formally the essential implicit function formula in the case of powers of expressions involving one or more variables. A modern reader will be struck by the absence of rules for the differentiation of transcendental functions. These were known at the time, but they had not been popularly formalized. L'Hospital had suggested to Bernoulli that he might wish to append to the *Analyse* his rules for the calculus of logarithms, but this was not done. However, the equivalent of these was implied by the well-known results on the area between the hyperbola and its asymptotes. The differentials of the trigonometric functions were at the time likewise geometrically inferred from the application of the characteristic triangle to the circle. The inverses offered no difficulty in view of the known formula for implicit functions.

The second section of the *Infiniment petits* (pp. 11–40) is devoted to problems of tangency. Inasmuch as the tangent is defined as the prolongation of an infinitely small side of the polygon which makes up the curved line, one sees immediately from the similarity of triangles that the subtangent is given by $y(dx/dy)$. Through this relationship, the author finds the tangents to the curves of his day: the higher hyperbolas and parabolas, the spirals of various kinds, the quadratrix of Dinostratus, the conchoid of Nicomedes, the cissoid of Diocles, the cycloid, the tractrix, and the curve of logarithms. Some of these curves are defined analytically, while others are described kinematically or in terms of geometrical properties. L'Hospital follows the Cartesian classification of curves into geometrical and mechanical, but inasmuch as no special notations are used for transcendental functions, only in the former category is "the nature of a curve" given by an ordinary equation.

The third chapter (pp. 41–54), on maxima and minima, opens with a definition of the basic terms: If the ordinate (appliquée) increases, as the abscissa (coupée) increases to a given value E, and thereafter decreases, then the ordinate at E is called the greatest; and similarly, *mutatis mutandis*, for the least ordinate. The treatment here differs from that in modern textbooks not only in being less rigorous, but also in that cusp maxima and minima are treated as extensively as extrema having horizontal tangents. L'Hospital points out that the differential of an increasing quantity is positive, that of a decreasing quantity is negative. Moreover, no quantity which varies continuously can go from positive to negative without going through zero or infinity. Hence to find maxima and minima of an algebraic expression, L'Hospital equates to zero both the numerator and the denominator of the differential, discarding results which lead to a contradiction. A few cases of applications in geometry and science are included. He does not explicitly state the various tests distinguishing maxima from minima, but these were well-known to the author and his times.

Points of inflection and of *rebroussement* (cusp points for which the tangents are horizontal) are treated in section four (pp. 55–70), and here the thorny question of second order differences arose. Leibniz had been unable to give a satisfactory definition of differentials of order greater than one, and the first attack on the calculus, launched by Nieuwentijdt in 1695–1696 while L'Hospital was preparing the *Analyse*, was centered on this inadequacy. In his preface (p. eiij) L'Hospital claimed that Leibniz had satisfactorily answered Nieuwentijdt's doubts about the existence of higher differentials, but this is belied by the author's own definition: "The infinitely small portion by which the difference of a variable quantity increases or decreases continually is called the difference of the difference of this quantity, or its second difference." That he obtained correct results in this connection is due less to his definition than to the pragmatic rule which follows it: "One takes as constant whichever difference one wishes, and treats the others as variable quantities." For example, the differential of the

function xy being $xdy+ydx$, the second differential is either $dd(xy)=xddy+2dxdy$ or $dd(xy)=yddx+2dxdy$. Here one sees the need to distinguish between the differentials of independent and dependent variables. This distinction was not at the time sufficiently emphasized, and even today numerous textbooks founder on the differential through neglect in this connection. L'Hospital then gives the geometrical description of points of inflection and *rebroussement* in terms of convexity and concavity, and he points out that these points are determined by making the second differential either zero or infinitely great. A more rigorous treatment is not, of course, to be expected of that period.

The concept of radius of curvature as such is not presented in the modern manner but is given in the context in which it arose historically. Huygens had been led by his pendulum clock to study the subject of evolutes and involutes (*developpées* and *developpants*), and this material is presented by L'Hospital in section five (pp. 71–103). Toward the close of this chapter, the longest of all, the author points out in a casual sort of way that $\overline{dx^2+dy^2}^{\frac{3}{2}}/-dxddy$ is a general expression for "the radius of an evolute." This is, of course, the equivalent of the modern formula for radius of curvature, where y is a function of x.

Sections six and seven (pp. 104–119, 120–130) are on caustics by reflection and refraction. These are not now traditional topics in elementary calculus, but they were popular new topics in the late seventeenth century. Moreover, L'Hospital and his master, Bernoulli, had shared with Tschirnhaus in their development. Section eight (pp. 131–144) is on envelopes of families of straight lines, and includes the well-known method of Leibniz of differentiating with respect to the parameter.

The portion of the book which in its day roused most discussion was section nine (pp. 145–163). This chapter carries the unenlightening heading, "Solution of some problems depending on the preceding methods," but it involves what would now be called indeterminate forms. The presentation is largely geometrical, but the basic result is the so-called "rule of L'Hospital." To find "the value" of a rational expression in x which for the abscissa in question takes the form $0/0$, he determines the ratio of the "differences" of the numerator and denominator for this abscissa. This rule appears not to have been familiar at the time, and so the author here might well have indicated the source of his material. The author never claimed the rule as his own, but when his friend Saurin implied that it was due to Leibniz, protest was raised by Jean Bernoulli that here, as elsewhere, L'Hospital had not given him due credit. Bernoulli's claim to the rule seems to be substantiated, and his name, rather than L'Hospital's, should be applied to it. But it should be pointed out that the traditional nomenclature is due to the fickleness of fortune rather than to any ill intent on the part of L'Hospital. The *Analyse des infiniment petits* occupies a place in the calculus somewhat analogous to that of Lavoisier's *Traité élémentaire* in chemistry almost a century later. Both books would have been more attractive had they included a substantial historical background, but in neither case should the omission of this be interpreted as an argument *e silencio* that the author claims credit for more than the general arrangement and exposition of the subject matter. One does not demand of writers of textbooks the same scrupulous regard for the citation of sources that one expects in the publication of the results of research work.

The tenth and last section of L'Hospital's treatise (pp. 164–181) is somewhat out of harmony with the remainder. It is a clever bit of salesmanship in which the author contrasts the elegant methods of the calculus with the awkward anticipatory procedures of Descartes and Hudde for determining maxima and minima. On the basis of this comparison L'Hospital emphasizes that the method of

Leibniz "gives general solutions, whereas the other furnishes only particular ones, and it extends to transcendental curves and it is not at all necessary to avoid incommensurables, which often is impracticable." This closing sentence of the first textbook on the calculus reminds one of the title of Leibniz's first paper on the new analysis a dozen years before: "A new method for maxima and minima, as well as tangents, which is not impeded by irrational quantities."

The *Analyse* of L'Hospital was composed two hundred and fifty years ago, and hence the material it contains does not in all cases parallel that which appears in modern textbooks. One fails to find, for example, the notions of function and limit. The first of these was in the air in L'Hospital's day, but it was popularized half a century later in Euler's *Introductio ad analysin infinitorum;* the latter was likewise known in a general sense through the calculus of Newton, but it was a hundred years before it became basic in the treatises of Cauchy. One misses also the now traditional work on Taylor's series. This, too, was known to James Gregory and Jean Bernoulli, but it was formalized after L'Hospital's time through Brook Taylor's work on finite and infinitesimal differences. Curve tracing is a further topic which is lacking in the *Analyse*. The seventeenth century here invented the analytical tools, but it was the eighteenth century which first applied them to the systematic study of a wide variety of curves. It is strange to relate also that L'Hospital's text is deficient in applications to science. Seventeenth century dynamics afforded a golden opportunity for apt and timely illustrations of the new calculus, but we are told in the preface (p. eij) that illness prevented the author from fulfilling his intention to add a section on applications to physics. The language and notation of the *Analyse* differ slightly from that now in use; there is an anachronistic emphasis on the ideas of ratio and proportion; and the material is divided into propositions, corollaries, and scholia. Nevertheless, a modern reader should have no difficulty in following the author's exposition.

The *Analyse des infiniment petits* appeared in a second edition at Paris in 1715. This was in reality a reissue of the original, but it contained numerous typographical errors. The work was reprinted the following year, and again in 1720. By this time the popularity of the calculus was such that a textbook on a lower level was desired. Publishers sought to capitalize on this desire and on the popularity of L'Hospital's work by providing commentaries. In 1721 Crousaz issued at Paris his *Commentaire sur l'analyse des infiniment petits*, consisting of an elementary introduction and a series of supplementary notes. This book was damned by faint praise on the part of Jean Bernoulli and met with severe criticism from Saurin. Four years later there appeared a posthumous and successful supplementary volume by Varignon, *Eclaircissemens sur l'analyse des infiniment petits*. The author appears to have had in mind the preparation of a new edition of L'Hospital's work, together with added notes. The publisher, however, felt that inasmuch as L'Hospital's *Analyse* already "had appeared several times in France and in foreign countries," and since "most men in the profession already had copies," he could save the public some expense by issuing Varignon's *Eclaircissemens* as a separate volume rather than as part of a new edition. This volume is on a much higher level than that of Crousaz, and although it follows L'Hospital section by section, it adds to this fresh problems and methods resulting from later research.

The rift between British and Continental mathematicians during the eighteenth century was not sufficiently wide to prevent Stone in 1730 from publishing an English translation of L'Hospital's *Infiniment petits*, ten years before Buffon reciprocally prepared a French translation of Newton's *Method of Fluxions*. The full

title of Stone's book is, *The method of fluxions both direct and inverse. The former being a translation from the celebrated Marquis de L'Hospital's Analyse des infiniments petits: and the latter supply'd by the translator.* The translator opens with praise of the author's work, "the Character whereof is so well establish'd," but adds that out of regard to Sir Isaac Newton he has altered some of the language and notation of the original to conform to the Newtonian. Increments or differentials, for example, are taken as fluxions, inasmuch as they are proportional to the latter; and these are denoted by $\dot{x}$, $\ddot{x}$, $\dddot{x}$, etc., instead of by dx, ddx, $dddx$, etc. A few years later Bishop Berkeley took full advantage, in the famous *Analyst* controversy of the confusion then existing in Britain between fluxions and infinitely small quantities.

Stone's translation, in part I, does not depart much from the original, and the diagrams are faithful reproductions of those of L'Hospital. Some new material is added, however, on logarithms, exponentials, and trigonometric functions. The second part or appendix (separately paginated) is on the inverse method of fluxions, and this constitutes a new work. L'Hospital apparently had hoped that his treatise would sometime be completed by a second part on the integral calculus. The pages of the *Analyse* (even in later editions) carry the heading *I. Part.*, but the author never completed another *partie.* Stone was but one of several who undertook to compose a suitable second half. His attempt was so successful that it was translated into French and published in 1735, "servant de suite aux infiniment petits de M. le Marquis de l'Hôpital."

The *Analyse des infiniment petits* continued throughout the eighteenth century to be the standard elementary manual on higher mathematics.[12] During the latter half of the century it found a worthy rival in the *Instituzioni analitiche* of Maria Agnesi (Milan, 1748), but frequency of publication indicates that the vogue for L'Hospital's work continued. It appeared in new French editions, revised and enlarged, in 1758 (Paris), 1768 (Avignon), and 1781 (Paris), and in Latin translations in 1764 and 1790 (both at Vienne).[13]

Toward the turn of the century, however, Lacroix[14] began the publication of his renowned series of textbooks, and one of these, the *Traité élémentaire de calcul différentiel et de calcul intégral,* (Paris, 1802) took the place during the nineteenth century which L'Hospital had held a hundred years earlier. The concepts of function and limit now took over the field which the infinitesimal "ghosts of departed quantities" had so long dominated. But the work of the first textbook on the calculus had been well done. The enthusiasm for the subject which it kindled and nourished has never abated, and modern textbooks are but new monuments to the influence of the *Analyse des infiniment petits.*

[12] The Abbé Sauri in his *Cours complèt de mathématiques* (5 vols., Paris, 1774), III, xij, refers to L'Hospital's *Analyse* as "connue de tout le mond."

[13] This paper does not pretend to give a complete listing of editions. I have examined those of 1696, 1715, 1730, 1768, and 1781, and the commentaries by Crousaz and Varignon, all of which are available at the New York Public Library or at Columbia University. Editions other than these have been listed on the basis of references found in standard sources, such as Poggendorff and the catalogues of the British Museum and the Bibliothèque Nationale. It is of interest to note that L'Hospital's *Traité analytique des sections coniques* likewise enjoyed a considerable popularity during the eighteenth century, appearing in editions of 1707, 1720, 1723 (English translation by Stone), 1770 (Venice), and 1776.

[14] S. F. Lacroix was perhaps the most prolific textbook writer of modern times, if allowance is made for multiple editions. In 1848 there appeared at Paris the 20th edition of his *Traité élémentaire d'arithmétique* and the 16th edition of his *Élémens de géométrie.* The 20th edition of his *Élémens d'algèbre* was published at Paris in 1858, and the ninth edition of the *Traité élémentaire de calcul* in 1881. And the above figures do not include the large number of editions in other languages.

Relations Inherent in the Gregorian Calendar

By THEODORE R. RUNNING
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1. *Introduction.* It is the purpose of this article to derive some of the relations inherent in the Gregorian calendar itself. An equation which is the algebraic representation of the calendar will be used. This equation is not new, but it is seldom used. In using the equation we must remember that the Gregorian calendar was not adopted in England and the colonies until 1752. It was introduced in Scotland in 1600.

2. *The equation.* The equation representing the Gregorian calendar is

$$\text{(A)}\qquad 7Q+R=D+Y+(Y-1)/4 -(Y-1)/100+(Y-1)/400$$

where Y represents the year and D the day of the year. Remainders in the right hand member of the equation are to be disregarded. The R represents the remainder after dividing the right hand member by 7. The last three terms take care of the extra day in leap years.

The question naturally arises, why the -1 in the numerators? The answer is, if we are considering a date in February of a leap year before the 29th the extra day is not to be counted, if a date later than and including the 29th the extra day is included in the value of D.

Equation (A) enables us to find the day of the week on which a particular date occurred. By its use we can also find the date corresponding to a given day of the week, provided the number of the week in the month, and the year are given.

The relation between R and the days of the week is given in Table I. The table is constructed so that the equation will be a correct representation of our calendar.

3. *Illustrations in the use of the equation.* *Illustrative Example 1.* Let it be required to find the day of the week on which July 4, 1904 occurred.

Proceed in the following way.

$$\begin{aligned} D &= 186 \\ Y &= 1904 \\ (Y-1)/4 &= 475 \quad \text{omit remainder} \\ (Y-1)/400 &= 4 \quad \text{omit remainder} \\ & \overline{2569} \\ (Y-1)/100 &= 19 \quad \text{omit remainder} \\ & \overline{7)2550} \\ & \overline{364} \quad R=2. \end{aligned}$$

TABLE I

R	Day of week
1	Sunday
2	Monday
3	Tuesday
4	Wednesday
5	Thursday
6	Friday
0	Saturday

From Table I it is seen that July 4, 1904 occurred on a Monday.

Illustrative Example 2. Find the date of the third Sunday in October 1919.

$$\begin{aligned} \text{For Oct. 1 } D &= 274 \\ Y &= 1919 \\ (Y-1)/4 &= 479 \\ (Y-1)/400 &= 4 \\ & \overline{2676} \\ (Y-1)/100 &= 19 \\ & \overline{7)2657} \\ & \overline{379} \quad R=4. \end{aligned}$$

Oct. 1, 1919 occurred on Wednesday, first Sunday on Oct. 5, third on Oct. 19.

Illustrative Example 3. On what day of the week did Nov. 22, 1800 occur? (Congress met in Washington for the first time.)

$D = 326$
$Y = 1800$
$(Y-1)/4 = 449$
$(Y-1)/400 = 4$
2579
$(Y-1)/100 = 17$
7)2562
366 $R = 0$, Saturday.

It is interesting to note that mistakes are sometimes made by writers in dating letters or other documents. The consistency or inconsistency of dates is easily determined by equation (A).

Among the many letters written by the poet Robert Burns, six were dated by adding the day of the week. In all the six he was inconsistent. In the letters written by the poet John Keats the dates were often given by adding the day of the week. In none of the dates was he found to be inconsistent.

It seems to be a general rule among historians to omit the day of the week in fixing the date of the occurrence of an event. Carlyle is one exception to the rule. In the "French Revolution" he records the dates of a considerable number of events by adding the day of the week. In one of these dates, however, he was in error. His statement is: "It is Monday, the 14th of September 1788: Rascality assembles anew, in great force, in the Place Dauphine." Monday was the 15th of September, the 14th was Sunday.

Would emphasis be added to the importance of a given event if the day of the week were given? For example, would it be more emphatic to write Thursday, July 4, 1776 instead of July 4, 1776? Or to write that the battle of Gettysburg was fought on Wednesday, Thursday, and Friday, July 1, 2, and 3, 1863 instead of giving only the days of the month and the year? This can be only a matter of personal opinion.

4. *The first day of a century.* The first day of a century can not occur on a Sunday, a Wednesday, or a Friday. This statement will now be justified. Let $Y = 100C+1$. $D = 1$. The right hand member of equation (A) becomes $1+100C+1+25C-C+C/4$. This reduces to $2+124C+C/4$. Taking out multiples of 7 we obtain the simple form $2+5C+C/4$. Substituting in this expression the values of C in Table II we obtain the corresponding values of R. From the table it becomes evident that no century can begin on a Sunday, a Wednesday, or a Friday.

TABLE II

C	R
$4n$	2 Monday
$4n+1$	0 Saturday
$4n+2$	5 Thursday
$4n+3$	3 Tuesday

5. *Centuries having identical calendars.* In order that centuries may have identical calendars throughout, it is evident that they must begin on the same day of the week. From Table II it is seen that between any two corresponding dates in such centuries four centuries intervene. Three of these four have 36524 days each, one has 36525 days. The difference then between any two corresponding dates in the two centuries will be 146097 days, an exact multiple of 7. This shows that any two corresponding dates of two centuries, with four centuries intervening, occur on the same day of the week.

The above may be stated in the following way. Let the years of any century be represented by $100C+y$ where y has values from 1 to 100.

Centuries in which $C = 4n$ have identical calendars throughout.

Centuries in which $C = 4n+1$ have identical calendars throughout.

Centuries in which $C = 4n+2$ have identical calendars throughout.

Centuries in which $C = 4n+3$ have identical calendars throughout.

6. *Recurring calendar years.* Each century contains a number of normal years with identical calendars, also a number of leap years with identical calendars. Such years will now be obtained.

TABLE III

First day of year	N. Y. = Normal years C	N. Y.			L. Y. = Leap years L. Y.	Group
Sunday	$4n$	06	17	23	12	
Friday	$4n+1$	34	45	51	40	I
Wednesday	$4n+2$	62	73	79	68	
Monday	$4n+3$	90			96	
Monday	$4n$	01	07	18	24	
Saturday	$4n+1$	29	35	46	52	II
Thursday	$4n+2$	57	63	74	80	
Tuesday	$4n+3$	85	91			
Tuesday	$4n$	02	13	19	08	
Sunday	$4n+1$	30	41	47	36	III
Friday	$4n+2$	58	69	75	64	
Wednesday	$4n+3$	86	97		92	
Wednesday	$4n$	03	14	25	20	
Monday	$4n+1$	31	42	53	48	IV
Saturday	$4n+2$	59	70	81	76	
Thursday	$4n+3$	87	98			
Thursday	$4n$	09	15	26	04	
Tuesday	$4n+1$	37	43	54	32	V
Sunday	$4n+2$	65	71	82	60	
Friday	$4n+3$	93	99		88	
Friday	$4n$	10	21	27	16	
Wednesday	$4n+1$	38	49	55	44	VI
Monday	$4n+2$	66	77	83	72	
Saturday	$4n+3$	94	(100)		(100)	
Saturday	$4n$	05	11	22	28	
Thursday	$4n+1$	33	39	50	56	VII
Tuesday	$4n+2$	61	67	78	84	
Sunday	$4n+3$	89	95			

Substituting the value $100C+y$ for Y in (A), letting $D=1$, and dropping $(y-1)/100$ and $(y-1)/400$ as remainders after dividing by 7 we obtain

(B) $7Q+R=1+5C+C/4+y+(y-1)/4.$

Let C take the values $4n$, $4n+1$, $4n+2$, $4n+3$, and for each of these values of C let y take the values from 1 to 100. Table III is obtained.

7. *Explanation of Table III.* At the top of the table the first group of fourteen numbers under N. Y. and L. Y. represent the years which begin on Sunday when $C=4n$. When $C=4n+1$ they begin on Friday, when $C=4n+2$ they begin on Wednesday, and when $C=4n+3$ they begin on Monday. The other groups are to be interpreted in a similar way. The normal years in each group have identical calendars; the leap years in each group have identical calendars. The year 100 is a leap year only when $C=4n+3$, for the other values of C it is a normal year.

All the normal years in group I when $C=4n$, in group III when $C=4n+1$, in group V when $C=4n+2$, and in group VII when $C=4n+3$ have identical calendars. All the leap years in these groups with the above values of C have identical calendars. It is seen that all the groups whose years begin on Monday, with proper values of C, have similar interpretation. In all there are seven such combinations of groups.

8. *Illustrations in the use of Table III.*

Illustrative Example 1. Let it be required to find the leap years in the 19th century in which February had five Sundays. The first day of February was a Sunday. Feb. 1 is the 31st day after the first day of the year, remainder 3. Let R represent the remainder corresponding to the first day of the years required. We

have the equation

$$R+3=1 \text{ or } 8,\ R=5,\ \text{Thursday.}$$

The years will then be found in the group in which the leap years begin on Thursday and $C=4n+2$. The years are 1824, 1852, and 1880. But the table gives all the leap years throughout the centuries in which February has five Sundays. They are the leap years of group II ($C=4n+2$), of group IV ($C=4n+3$), of group V ($C=4n$), and of group VII ($C=4n+1$).

Illustrative Example 2. Let it be required to find the years in which Christmas occurs on Wednesday. In normal years Dec. 25 is 358 days after the first day of the year, remainder 1. Therefore

$$R+1=4,\ R=3,\ \text{Tuesday.}$$

The normal years in which Christmas occurs on Wednesday begin on Tuesday. The normal years are given in group II ($C=4n+3$), in group III ($C=4n$), in group V ($C=4n+1$), and in group VII ($C=4n+2$).

In leap years Dec. 25 is the 359th day after the first day of the year, remainder 2 This gives the equation

$$R+2=4,\ R=2,\ \text{Monday.}$$

The leap years in which Christmas occurs on Wednesday begin on a Monday They are found in group I ($C=4n+3$), in group II ($C=4n$), in group IV ($C=4n+1$), and in group VI ($C=4n+2$).

Eighty-Second Regular Meeting of the Association of Mathematics Teachers of New Jersey, Rutgers University

May 4, 1946

PROGRAM

Problems in the Teaching of Mathematics in New Jersey

DR. FRED L. BEDFORD, Presiding

10:00 A.M. Report from the National Policy Commission
Dr. Virgil S. Mallory, State Teachers College, Montclair

10:10 A.M. Problems in the Elementary School
Dr. Amanda Loughren, Supervisor of Mathematics, Elizabeth

10:25 A.M. Problems in the Junior High School
Miss Mary C. Rogers, Roosevelt Junior High School, Westfield

10:40 A.M. Problems in Preparatory Mathematics in the Senior High School
Mr. J. Dwight Dougherty, Eastside High School, Paterson

10:55 A.M. Problems in General Mathematics in the Senior High School
Mr. Chauncey W. Oakley, Manasquan High School, Manasquan

11:10 A.M. Presidential Address: A Forward Look on Mathematical Education in New Jersey
Dr. Fred L. Bedford

11:40 A.M. Discussion of Problems
Dr. Howard F. Fehr, State Teachers College, Montclair

12:00 M. Business Meeting

Membership Report	Miss Margaret Aeschbach
Treasurer's Report	Miss Mary C. Rogers
Report of the Nominating Committee	Mr. J. Dwight Dougherty

Election and Presentation of Officers for 1946–1947

1:00 P.M. Luncheon in the University Gymnasium.

Effective Guidance in Problem Solving

By HARRY KARSTENS
Lyons and Carnahan, Chicago, Ill.

THE AMOUNT of research in the field of problem solving, while not in proportion to the recognized importance of the subject, is still far from meagre. Unfortunately, for several reasons the results in some instances have been contradictory. Then, too, the published writings of prominent workers in the field have included assertions and recommendations that were not, at times, in accord with each other.

It has been difficult for in-service teachers to properly evaluate the numerous techniques and methods that have been set forth in texts and in professional literature. Effective guidance in problem solving requires a clear concept, on the teacher's part, of the function of problems in arithmetic and a knowledge of what problem-solving techniques are sound and why and, by the same token, what techniques are unsound and why.

The teaching of problem solving, which makes greater demands on the ability and resourcefulness of a teacher than any other phase of arithmetic, is a particularly delicate and exacting procedure when handling low-ability groups. While intelligence is undoubtedly a major factor in solving problems, children with low I.Q.'s can nevertheless be guided in their thinking. Even though a dull child can never attain more than limited success in problem solving, every child can be definitely helped. The successful teacher limits her expectations in individual cases and concentrates on skillful and sympathetic guidance.

The first requirement for teaching problem solving is a clear concept of the function of problems. There is no such thing as determining the relative importance of problems and computation since the two are inseparable; either is valueless without the other. Numbers in life are always concrete; computation in life is always undertaken to solve a problem. Computation in arithmetic achieves its objective in problem solving. Computation, in itself, is meaningless; problem solving, without computational ability, is futile. Computation makes problem solving possible; problems give sense to computation. *Problems are the humanizing element in arithmetic.* Unless problems and computation run *concurrently* the "spark" is likely to disappear from arithmetic work.

What factors must be taken into consideration *before* attempting actual guidance in problem solving? A pupil's reading ability, his I.Q., his arithmetical background and his social background must all be taken into consideration. A pupil must be able to read the problem; his I.Q. must be adequate for the reasoning involved; his experience with arithmetic terms and processes must be adequate for the computation in the problem; and his social background must enable him to understand the problem situation.

The solution of a written problem necessarily entails the ability to read the problem. The teacher should see to it that any arithmetic vocabulary that appears in a problem is already familiar to the child. The solution of a problem should not include the learning of a new arithmetic term.

A child deficient in reading ability, particularly in study type reading, can do little else than manipulate the numbers in an effort to arrive at some sort of answer. Occasional fortuitous answers only serve to conceal his failure to grasp the problem situation. The most helpful training for such a child is through special study type reading exercises. Reading to reproduce is perhaps the most important skill

to stress. The primary purpose in reading a problem is to "find out what it is all about." The pupil should be able to retell the problem situation in his own words. Gradually the ability to distinguish essential from non-essential sentences or phrases is acquired, so that in the retelling the pupil gives only the gist of the situation. At this point, the method of solution usually becomes obvious; the pupil selects the facts he needs and does the computation. The training in study type reading might well be grouped under these headings:

1. Varied reading for background, sentence structure and comprehension.
2. Special exercises in reading to reproduce.
3. Specific training in selecting the essential elements (not necessarily the numbers) in a situation.
4. When the training in 2 or 3 involves the reading of problem material, a clear understanding of the problem question must precede the selection of essential elements.

Note that the selection of needed *facts* comes *after* the planning of the solution. The planning comes after the recognition of the essential phrases or sentences. The delay in the selection of the needed numbers seems particularly wise when we realize that one major cause of failure, as borne out in Banting's study is "The habit of focusing the attention upon the numbers and being guided by them instead of by the conditions of the problem." Children are prone to plunge into the solving of a problem without understanding the situation and without realizing what is asked for.

A pupil must have adequate background, both social and arithmetical, else he is not ready for that problem. The necessary computational skills must have been adequately developed. His number concepts and his process understandings must be rich enough to enable him to see the number relationships in the problem with not more than a reasonable degree of effort. His social background should be such that he can understand what is happening in the problem. In this connection it should be noted that a poor reader will often misunderstand the conditions in a problem.

The term "teaching problem solving" carries unfortunate implications. "Teaching problem solving" implies formal methods and set routines while the term "guidance in problem solving" connotes a sympathetic molding of the pupil's thinking and an informal directing of his work habits. The problem solving material in many texts and courses of study is based primarily on a pattern of formal analysis in which the pupil asks himself three questions.

1. What am I to find out?
2. What facts (numbers) are given?
3. What shall I do with the numbers?

The number of steps is often increased but the basic order remains the same. An investigation by Washburne and Osborne indicated that the method of formal analysis had no appreciable effect upon the ability of pupils, especially the brighter pupils, to solve problems. The difficulty seemed to be in bridging the gap from step 2 to step 3. Since Washburne and Osburne got better results by simply leaving the pupils to their own devices in solving problems, there apparently was some aspect of the method that confused instead of clarified.

In the first place, any method of formal analysis tends to stress techniques instead of understandings. Children have little difficulty in solving problems of their own making *because they thoroughly understand the situation.* Formal analysis emphasizes a set routine and tends to short cut thinking. Furthermore, instead of setting a logical plan of attack as exhibited in a good method of analysis as a desirable, *ultimate* objective of a problem-solving program, a formal pattern is imposed upon the children before they are ready for it.

In the second place, *however* a pattern

is used and *whenever* it is presented, the steps (after reading the problem) might better be as follows:

1. What is the problem about?
2. What am I to find out?
3. What are the essential phrases or sentences?
4. What must I do to solve the problem?
5. What facts will I need?

After this, the computation, checking, and evaluation of the answer should follow in due order.

Note that in step 1. the emphasis is upon the *meaning* of the problem, *aside from its arithmetical implications.* Step 3 has to do with the arithmetical implications *but is not particularly concerned with the actual numbers.* The numbers themselves are not in the foreground until step 5. In planning the solution, *meaning* should be stressed, not manipulative techniques. Manipulation is essential in arriving at the answer, but meaning is essential in determining the manipulation *and must precede manipulation.*

Insistence upon formal analysis, especially in the lower grades, is likely to have a demoralizing effect on many pupils and many times results in mechanical, meaningless performances. Overman says, "A pupil may recite very glibly a formal analysis such as the following: 'If one pencil costs 4¢, three pencils will cost three times as much as one pencil, or 3 times 4¢ which is 12¢,' but the chances are that the thinking back of the analysis is no more his own than is the phraseology in which it is expressed. Pupils should be permitted to explain or analyze a problem in their own way and in their own words." A good teacher will determine the "thinking level" of a child and then encourage him to refine his reasoning and lead him into more economical ways of thinking.

Many have tried the "method of analogies" in connection with difficult problems. In this method a difficult problem is introduced through a similar but simpler problem with the hope that the pupil will see the analogy and be able to solve the difficult problem. Some writers have even attempted the wholesale grouping of problems according to their characteristics. Such a procedure can have scant hope of success for at least two good reasons. First, as Forban points out, the use of this method may result in the habit of associating problems with a certain type problem merely because there is a superficial resemblance. This does not stimulate thinking but may well discourage thinking. Second, identifying problems, especially life problems, with a certain type problem is usually a difficult and well-nigh impossible procedure. Since real life problems occur in endless variety, the grouping of problems according to structural patterns is not effective. On the contrary, pupils should learn how to use the same technique for a variety of problems and to use different techniques for similar problems. Pupils should never become dependent upon a stereotyped form of expression in problems. Neither should solutions be stereotyped. Any solution that is logical and results in a correct answer should be marked correct, even though the method was not the best. As a matter of fact, pupils should be led to see multiple solutions and to judge their relative merits.

Making a preliminary estimate is regarded by many as a practical way to prevent absurd answers. In certain situations estimating is a real help but in other situations estimating is quite difficult, if not impossible. An estimate that is as difficult and as subject to errors as the actual computation is valueless. For instance, in a situation involving several purchases at a grocery, the mental computation required in estimating the amount of change would in most cases be much more difficult than the actual written computation. Furthermore, if the written answer was wrong but was also close to the estimate, the pupil would be misled (*by the estimate*) into thinking his answer was correct.

The principal value of estimating is in the field of very large numbers and in decimals. In a tax problem involving $1,134,576 \div \$197,342,680$, a helpful estimate may easily be made by thinking "1 million divided by 197 million equals about $1 \div 200$, or $\frac{1}{2}$ of 1%." In decimals, $14.135 \div 1.943$ may be thought of as "$14 \div 2$, or about 7." To inject estimating into a situation where estimating is impractical is confusing and discouraging.

Much emphasis is placed by some on the use of "cues" in problem solving wherein the pupil takes his cue from the wording of a problem. To illustrate, Morton cites the case of a pupil who, when asked how he knew he should multiply in solving a problem, said, "Whenever it says 'at,' I multiply." When resort is made to cues, either the problem situation is far removed from the pupil's experience or an attempt is being made to eliminate the need for thinking or reasoning. Wilson and Stone characterize the teaching of cues as "a process which eliminates the need of reflective thinking." Even though the lack of understanding inherent in the "cue method" be disregarded, the learning of a sufficient number of cues proves to be an impossible task. When it is pointed out that in 1-step problems alone, Judd found 1900 ways of expressing the fundamental operation, it should be realized that utilizing the form of the problem question as a clue to the operation is a futile undertaking.

The importance of understanding in problem solving makes reflective thinking an outstanding factor. Osborne's tabulation of 30,000 errors in problem solving showed that 40% of the errors were due to lack of understanding, evidenced by failure to grasp the problem situation and by inability to see quantitative relationships. In teaching the fundamental processes, it is extremely important to show how each phase of computation functions in concrete situations. Teaching a pupil to subtract accurately is only part of the job. The pupil must realize all the possibilities of subtraction through a wide variety of concrete situations involving subtraction. The development of rich fundamental concepts bears abundant fruit in problem solving. In fact, such development is an integral part of the problem solving program.

It should be evident from the foregoing that *meaning is all important in a guidance program in problem solving.* The teacher's problem is to take stock of each individual's "mental tools," *such as they are,* and guide him into more efficient and more mature ways of using those tools. Her efforts should be devoted toward making each child a *thinking* individual. Since *all other phases of arithmetic culminate in the problem solving program,* guidance in problem solving should be the most interesting and at the same time the most challenging phase of arithmetic teaching.

Algebra

Algebra is the course
  Where the teacher talks until she's hoarse,
Trying to teach us the right way
  Which will take her many a day.
We study until we are blue,
  Finding out what X is equal to.
The problems are plenty tough
  But the teacher says she didn't give us enough.
We study all day and study all night,
  But still our problem doesn't come out right.
To biology some are true;
  But I'm for algebra through and through.

Algebra is a good subject, as you may learn;
  But if you don't understand it, it's not worth a durn
Algebra is super
  So be a good trooper;
Take a tip from me
  And learn it to a tee.

Written by Bud Nesbitt
Clovis Senior High School
Submitted by Marjorie Williams
Department of Mathematics
Clovis, New Mexico

Concerning the Irrationality of $\sqrt{2}$

By Wesley J. Lyda
Morgan State College, Baltimore 12, Md.

In the November (1945) issue of Mathematics Teacher, Howard Eves presented in an article under the title—Irrationality of $\sqrt{2}$—a number of interesting proofs of the principle, no rational number squared is equal to 2. In this paper, the writer proposes to present proofs of two corollaries to this principle; namely:

1. *For any number whose square is less than 2, one can always find a second number greater than the given number whose square is also less than 2.*

2. *For any number whose square is greater than 2, one can always find a smaller number whose square will also be greater than 2.*

Proof of Corollary 1: For any number whose square is less than 2, one can always find a second number greater than the given number whose square is also less than 2.

Let p and q be positive and relatively prime to each other.

Let $\left(\frac{p}{q}\right)^2$ be less than 2.

$$\left(\frac{p}{q}\right)^2 < 2.$$

Take another number which is larger than $\frac{p}{q}$; hence, its square is larger than

$$\left(\frac{p}{q}\right)^2.$$

Let this number be $\left(\frac{p}{q}+\frac{1}{10^n}\right)$

$$\left(\frac{p}{q}+\frac{1}{10^n}\right) > \frac{p}{q}$$

$$\left(\frac{p}{q}+\frac{1}{10^n}\right)^2 > \left(\frac{p^2}{q^2}\right).$$

To prove: $\left(\frac{p}{q}+\frac{1}{10n}\right)^2 < 2$

$$\left(\frac{p}{q}+\frac{1}{10n}\right)^2 = \frac{p^2}{q^2} + 2\,\frac{p}{q}\cdot\frac{1}{10n} + \frac{1}{10^2n}.$$

Adding and subtracting 2 from the right member of the identity, one obtains:

$$\left(\frac{p}{q}+\frac{1}{10n}\right)^2 = 2 - \left[2 - \frac{p^2}{q^2} - \frac{1}{10n}\left(2\,\frac{p}{q}+\frac{1}{10n}\right)\right].$$

Examining the expressions of the right member of the above identity, one notes the following concerning:

1. The expression—$2-\frac{p^2}{q^2}$; it is positive because $\frac{p^2}{q^2}$ is positive and is less than 2. Let us call the difference—$2-\frac{p^2}{q^2} \cdot k_1$.

2. The expression—$\frac{1}{10n}\left[2\,\frac{p}{q}+\frac{1}{10n}\right]$; one can make n as large as he desires. n is positive; therefore, $\frac{1}{10n}\left(2\,\frac{p}{q}+\frac{1}{10n}\right)$ can be made as small as we wish but it is always less than k_1. Let us call the value of $\frac{1}{10n}\left(2\,\frac{p}{q}+\frac{1}{10n}\right)$ equal to k_2. Hence,

$$\left(\frac{p}{q}+\frac{1}{10n}\right)^2 = 2 - [k_1 - k_2] \quad k_1 > k_2$$

$\left(\frac{p}{q}+\frac{1}{10n}\right)^2 = 2-K$; let $K = k_1 - k_2$, K being a positive number.

$\left(\frac{p}{q}+\frac{1}{10n}\right)^2$ is equal to a number less than 2.

Conclusions:

$$\frac{p^2}{q^2}<2;\quad \left(\frac{p}{q}+\frac{1}{10n}\right)^2>\frac{p^2}{q^2};\ \text{that is,}$$

$$2>\left(\frac{p}{q}+\frac{1}{10n}\right)^2>\frac{p^2}{q^2}$$

$$\left(\frac{p}{q}\right)^2<\left(\frac{p}{q}+\frac{1}{10n}\right)^2<2.$$

Proof of Corollary 2: For any number whose square is greater than 2, one can always find a smaller number whose square will also be greater than 2.

Let c and d be positive and relatively prime to each other.

Let $\frac{c^2}{d^2}>2$

$$\left(\frac{c}{d}-\frac{1}{10n}\right)<\frac{c}{d}$$

$$\left(\frac{c}{d}-\frac{1}{10n}\right)^2<\frac{c^2}{d^2}.$$

To prove $\left(\frac{c}{d}-\frac{1}{10n}\right)^2>2$

$$\left(\frac{c}{d}-\frac{1}{10n}\right)^2=\frac{c^2}{d^2}-2\,\frac{c}{d}\cdot\frac{1}{10n}+\frac{1}{10^2n}.$$

Adding and subtracting 2 to both members.

$$\left[\frac{c}{d}-\frac{1}{10n}\right]^2$$

$$=2+\left[\frac{c^2}{d^2}-2-2\,\frac{c}{d}\cdot\frac{1}{10n}+\frac{1}{10^2n}\right]$$

$$\left[\frac{c}{d}-\frac{1}{10n}\right]^2$$

$$=2+\left[\frac{c^2}{d^2}-2-\frac{1}{10n}\left(2\,\frac{c}{d}-\frac{1}{10n}\right)\right].$$

Examining the expressions:

(1) $\frac{c^2}{d^2}-2$; ones notes that $\frac{c^2}{d^2}-2$ is positive.

(2) $\frac{1}{10n}\left(2\,\frac{c}{d}-\frac{1}{10n}\right)$, n, which is positive, can be made as large as you desire; hence, this expression can be made as small as you like, approaching zero as a limit.

Let $r_1=\frac{c^2}{d^2}-2$, a positive number.

Let $r_2=\frac{1}{10n}\left[2\,\frac{c}{d}-\frac{1}{10n}\right]$;

$$r_2=\frac{1}{10n}\left[2\,\frac{c}{d}-\frac{1}{10n}\right]<\frac{c^2}{d^2}-2.$$

That is, $r_1>r_2$.

Let $R=r_1-r_2$

$$\left(\frac{c}{d}-\frac{1}{10n}\right)^2=2+(r_1-r_2).$$

Hence $\left(\frac{c}{d}-\frac{1}{10n}\right)^2=2+R$

$$\left(\frac{c}{d}-\frac{1}{10n}\right)^2>2;$$

$$\left(\frac{c}{d}\right)^2>\left(\frac{c}{d}-\frac{1}{10n}\right)^2>2.$$

Yearbook Supply Dwindling

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Division of Decimal Fractions

By MARY E. CROFTS
Buffalo, N. Y.

THE ARTICLES on division of decimal fractions in the February and May 1945 issues of THE MATHEMATICS TEACHER invite this paper.

On page 220 of our Sixteenth Year Book, Dr. Buckingham states: "They shift decimal points without knowing why." Challenged on my agreement with this statement, by one of the principals, who said that she was sure her boys and girls understood, I proposed the following test.

a. Divide 10.556 by 2.03.
b. Justify moving the decimal point.

Lest the students be confused by the phraseology of (b), I asked the teachers to state (b) in the phraseology to which the children were accustomed. The answers to part (b) yielded the following tabulation:

22 said: "to make the divisor a whole number."
23 said: "The division could not be done unless the point were moved."
2 said: "It is easier"
6 left the space blank
6 had erroneous ideas
2 had the idea that dividend and divisor may be cast into fraction form.

The six errors centered on the belief that changing the position of the point does not change the value of the number.

With this small study as the basis for attack, I asked for 20 teachers to volunteer to teach decimal division as the inverse of multiplication. Seventy-three responded. These volunteers had two serious handicaps: they themselves had been trained on the caret system; the children had had one year of practice with it. Their opinions follow:

56 carried both methods, synchronously, and preferred the inverse method.
13 found the children confused and unable to change.
4 disliked the inverse method.

Those who liked the inverse method said that: it is more easily taught; it is more logical; it is more convincing; it is more easily recalled; it yields a higher percentage of accuracy. Those who disliked the inverse method said that the caret method is easier.

Two reasons may be given for the lack of understanding of the caret method. First: the textbooks give no adequate explanation. Second: the method is a device. Three years ago, with adoption in view, we examined arithmetics of fifteen authors. Twelve of them gave no explanation of the principle underlying the shift of points. Instead of explanation these books had imperative mood directions—"place," "move," "move." Three authors had the correct explanation, suitably illustrated. One of these three gave this soul-scarifying addendum: "But why do all this thinking. Just place—and move—and move." I wrote to one book company asking why the author had substituted the imperative mood for an explanation and the answer came from the author who said that he considered the explanation too difficult for sixth grade children to understand; so he had put it into the "Manual for Teachers."

That the caret method is a "device" is so stated by Dr. Morton in his "The Teaching of Arithmetic" vol. II, page 356. Addition and subtraction of integers, common fractions, and decimal fractions are always associated as inverse operations. Multiplication and division of integers and common fractions are always so associated. It is only in the division of decimal fractions that texts for thirty years have "devised" against the relationship. Why?

One of the teachers, reporting in my experiment, told of a former student who was preparing for a marine examination and who had asked her to help him with division of decimal fractions. Said he: "If it had ever been associated with anything, I think that I could recover it with the aid of the book; but the book gives no help."

Supported by Dr. Morton's opinion that "the best approach seems to be through the medium of the common fraction," Buffalo Public Schools have changed from the caret to the inverse method. At the end of the year 1944–1945, teachers who had taught the caret method prior to our change were asked to evaluate both methods as to teaching, learning, and recalling. Thirteen preferred the caret method. In its favor they listed the immediate placing of the decimal point and the mechanical nature of the process. One report said: "Slow children can get the answer even if they do not understand the process." Against the inverse method, they listed the difficulty with zero when the dividend is integral and the divisor a decimal fraction. Seventy preferred the inverse method. In its favor they listed the inverse relation which makes for ready understanding of the position of the decimal point, and the lack of preparatory movement. Against the caret method they listed its isolation and the difficulty of the prefixed zero in the quotient. Three had no preference. In the matter of recall, eleven found the caret method the better while thirty-two supported the inverse method. Forty-three had varying modifications of these findings, being in agreement, generally, that recall varies directly as original understanding.

My test is not "scientific" as Mr. Johnson proposed that a test should be. It merely served as the pivot on which we turned from the caret to the inverse method.

We still have two difficulties. Helpful parents and transfers.

If Mr. Johnson and Mr. Brown read "How Green Was My Valley," they probably had antipodal emotion over the author's recognition of the caret method in these two remarks: "and the decimal points move up and down like monkeys," "and the look on my mother's face when the decimal point started its travels up and down the line was something to see."

With Mr. Brown, I should rejoice to have texts and school systems revert to the inverse treatment of decimal division.

Mathematics in Nature

By MERTON TAYLOR GOODRICH
Keene Teachers College

Year after year, comes winter then spring,
Flowers appear and you count if you will
  Petals, or sepals, or stamens.
This is arithmetic's easiest skill,
Yet it is found in each beautiful thing.

Butterflies pause for a moment to rest,
Symmetry perfectly made by their wings,
  Opening wide in the sunshine.
Birds fly in curves. Note how each as it sings
Traces with music a graph at its best.

Long before men ever pondered the sight,
Raindrops fell downward in parallel lines—
  Spheres into circles were broken;
While in the winter with varied designs,
Floated the soft little hexagons white.

Night after night and wherever you roam,
Every star has its place where it shines,
  Making a pattern of beauty—
Triangles, trapezoids, pentagons, lines—
Nature's geometry, vast, in the dome.

Infinite space—and thru infinite time,
Formulas, integrals, always were there—
  Masters discover them slowly.
More will be found in the earth and in air.
God is the architect. Truth is sublime.

Can We Prove 2 Plus 2 Equals 4?

By LEE BYRNE
Elgin Academy, Elgin, Ill.

As REPORTS are circulating about one campus that no mathematician can prove that 2 plus 2 equals 4, it may be worth while to correct this impression. There are various known ways of formulating the foundations of arithmetic. No one of these formulations would be regarded as satisfactory, unless facts of addition could be proved in its terms, and "2 plus 2 equals 4" could serve as well as anything else to illustrate such a proof.

Perhaps the fifty-year old formulation of arithmetic due to an Italian mathematician named Peano provides means for such a proof that would come as near to being intelligible to a non-mathematical reader as any that could be selected, though it should be needless to say that the validity of a mathematical proof does not depend on whether it is either convincing or intelligible to one who has not engaged sufficiently in related mathematical study.

Proofs are not created out of nothing. In most cases use is made of definitions which in the end must have started with some undefined terms. More directly use is likely to be made of previously proved assertions, which in the end go back to some accepted but unproved assertions called postulates. In this respect a mathematician has a choice, however. He can start with certain mathematical undefined terms and (or) certain mathematical unproved assertions, or he can avoid either or both of these courses by constructing his mathematical definitions out of the concepts already present in modern logic, and deriving his mathematical assertions by means of modern logic from the truths already explicitly stated in modern logic. (On the last-mentioned point, exception would have to be made in the case of one or two very advanced mathematical ideas.)

Proving mathematical truths in general without employment of mathematical undefined terms or mathematical postulates is a longer and more difficult program than starting freely with selected mathematical undefined terms and mathematical postulates. Peano uses both, and that is one reason why a proof in his terms is likely to be more nearly intelligible to readers of limited mathematical preparation.

In setting up arithmetic Peano uses as undefined mathematical terms the following: "number," "one," "successor," and also takes for granted the properties of a relation "equals" or "is equal to." He uses about five mathematical postulates, which need not be given here, except to remark that one of them is "one is a number," and another "every number has a successor, which is also a number." From the context it becomes clear that Peano's undefined "numbers" will be whole numbers and positive.

For the proof to be presented here there is need of prior definition of the numbers "two" and "four," and in fact also "three," and prior definition of the function "plus," that is, in each case definitions sufficient for the purpose (and for the further development of arithmetic).

Definition of 2: 2 is (defined as) equal to the successor of 1

Definition of 3: 3 is (defined as) equal to the successor of 2

Definition of 4: 4 is (defined as) equal to the successor of 3

Suitable definition of "plus" is accomplished by two defining statements, an example of so-called "inductive" definition and also "definition in use":

Definition of plus:

(1) If m is any number, m plus 1 is (defined as) equal to the successor of m

(2) If m and n are any numbers, m plus the successor of n is (defined as) equal to the successor of (m plus n)

The following proof would then satisfy any mathematician technically acquainted with the foundations of arithmetic. Theorem. 2 plus 2 is equal to 4. Proof:

(a) 2 is equal to the successor of 1. (By Def. of 2)
(b) 2 plus 2 is equal to 2 plus the successor of 1. (From (a))
(c) 2 plus 2 is equal to the successor of (2 plus 1). (From (b) by second part of Def. of plus)
(d) 2 plus 1 is equal to the successor of 2. (By Def. of plus)
(e) The successor of 2 is equal to 3. (By Def. of 3)
(f) 2 plus 1 is equal to 3. (From (d) (e))
(g) 2 plus 2 is equal to the successor of 3. (From (c) by (f))
(h) The successor of 3 is equal to 4. (By Def. of 4)
(i) 2 plus 2 is equal to 4. (From (g) by (h))

Statements (a) to (i) constitute the proof, without the remarks added in parentheses; the latter merely assist a reader to recognize the relations. Moreover, (a), (e) and (h) only repeat what is already stated in the corresponding definitions, and could be omitted when those definitions have already been given.

The proof above depends on the principles of modern logic for the relation of logical consequence between each statement and statements which have preceded. The same would be true of any proof. But in a more adequate account one, two, three, four and plus and also this kind of number could all be defined in terms of the concepts of modern logic alone, while "successor" would not be needed. Nor would any specifically mathematical postulates be required, the only unproved assertions used being those of a modern logic which is presupposed in scientific work generally. One advantage of such an account would be that a clear distinction would emerge between the cases in which a two and a two make a four and those cases in which a two and a two do not make a four. The mathematical "two plus two" has to exclude the very real instances of the latter kind. But a proof in these more ambitious terms would be longer and more intricate, and would call for a great deal of preparatory explanation.

Algebra

Now English isn't hard to take,
  You read and know what's there,
You study speech for grammar's sake;
  It's not too hard to bear.

There're other subjects too; I take
  Biology and art.
They're easy if you work and make
  The teachers feel you're smart.

But algebra, heaven forbid!
  I must have been stark mad;
It's Greek to me; I never did
  Learn even how to add.

X's and Y's and Z's run through
  My memory until
This is the only thing that's true,
  "X is unknown still."

Written by Robert Grossen
Clovis Senior High School
Submitted by Marjorie Williams
Department of Mathematics
Clovis, New Mexico

We Recommend the Teaching of Statistics in High School

By Mrs. Ollie Sanders Smith *and Others*[1]
Hartsville, South Carolina

Inasmuch as statistical thinking is involved in practically every phase of modern living, we believe everyone should be familiar with the modes of thought which it affords.

The study of statistics offers unlimited opportunities for applications of mathematics in fields where social problems exist; business, industry, government, agriculture, medicine, psychology, sociology, and education are enormously rich in problems, data, and opportunities.

Statistics enters into the legislative, managerial, and evaluative aspects of all large undertakings. Surveys, predictions, analyses, norms, testing, and quality control are constantly required. Newspaper readers are confronted with data, terms, concepts, tables, and graphs which are of statistical origin.

The statistical way of thinking offers large cultural advantages to everyone through adding greater perspective to his considerations, reducing the tendency toward dichotomous thinking, and developing the disposition to examine evidence critically.

Vocational opportunities for statistical workers on all levels have increaded tremendously in recent years. Accordingly, the schools (1) should recognize their responsibility for giving preliminary training on suitable levels to help meet the growing need, and (2) they should provide adequate exploratory contacts with statistical procedures as a means of guiding interested young people into a large occupational field.

In view of these and other considerations *it is our recommendation* (1) that at least some statistical training be given to all high school students, and (2) that further opportunities for broadening experiences with statistical points of view and techniques be made available for those high school students who wish them.

This recommendation is in conformity with a large proportion of the items in the Check List of "functional competence in mathematics" (Thesis 1) appearing in the "Second Report of the Commission on Post-War Plans,"[2] notably, items 4 (tables); 5 (rounded numbers); 7 (collecting and interpreting data; using averages; making and interpreting graphs); 12 (measurement, errors, approximations); 13 (scales); 14 (scale drawings); 19 (substituting in formulas); 26 (personal affairs, home, and community); 27 (communication, travel, transportation); and 28 (consumer problems; insurance).

For instructional purposes, available data are inexhaustible, problem material is abundant, and textual aids are adequate for a start.

Technical concepts needed for pratical statistical work are few, and the analytical skills called for are those already familiar to mathematics teachers.

[1] Report of Committee on Collecting and Analyzing Data, of the Fifth Institute for Teachers of Mathematics, Duke University, Durham, N. C., July 1945. Mrs. Ollie Sanders Smith, Hartsville High School, S. Car., Chairman; Elizabeth Churchill, Columbia School, Rochester, N. Y.; Frances C. Johnson, Oneonta High School, New York; W. W. Rankin, Mathematics Department, Duke University; Douglas E. Scates, Education Department, Duke University; Veryl Schult, Wilson Teachers College, Washington, D. C.; Ernest C. Thayer, Orleans Parish Schools, New Orleans, La.; Naomi Wharton, Central High School, Richfield Springs, N. Y.; and James H. Zant, Mathematics Department, Oklahoma A. and M. College, Stillwater, Okla.

[2] The Mathematics Teacher, 38: 195–221, May 1945.

A working vocabulary may consist of some half-dozen new terms. The preparation for teaching statistics might be obtained through a short workshop period, or through independent study of simple texts. Further training, through a regular course in statistics, is valuable for broadening the teacher's resourcefulness and adding to his confidence.

In high school courses the laboratory method (including gathering of data for a given purpose, and the preparing of reports) should be emphasized in contradistinction to more theoretical, abstract processes. The "general feeling" for the statistical approach to social problem solving, rather than technical manipulative skills, is the principal goal to be sought.

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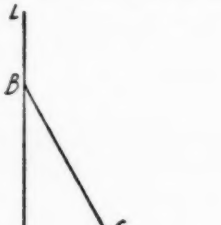
◆ THE ART OF TEACHING ◆

Discovering the Tangent

By BERTHA WEIR PALMER
Junior High School, Glen Rock, N. J.

TANGENTS, sines and cosines furnish an excellent opportunity for creative learning for junior high school pupils. They also furnish the pupils an opportunity to have the thrill that comes to a discoverer if they are taught by the laboratory method.

The class can be divided into groups of three or four pupils to a group. Let the first pupil in each group draw a base line CA one inch long. Have the second pupils draw theirs two inches long, the third pupils three inches long and the fourth pupils four inches long. At the end C of the base line have them all erect with a



protractor a line CL perpendicular to the base line. Have the right angle at C labeled 90°. Now give each group of four an acute angle to be made at point A. These might be 30°, 45°, 60° and 64°. The pupils should extend the line which form these angles to the perpendicular line CL at B and form a right triangle. Now the line CB should be labeled side opposite angle A. The side CA should be labeled the side adjoining angle A. The pupils should measure each of these lines very carefully and write the dimensions on their figures. A ruler graduated in tenths is convenient to use.

Each pupil should set up this form with the data from his right triangle.

$$\frac{\text{Side opposite } \angle 60^\circ}{\text{Side adjoining } \angle 60^\circ} = \frac{AB}{CA} = \frac{3.46}{2} = 1.73$$

Use the blackboard to set up this form for each group of angles.

$$\frac{\text{Side opposite } \angle 60^\circ}{\text{Side adjoining } \angle 60^\circ} = \frac{AB}{CA} = \frac{1.7}{1} = 1.7$$

$$\frac{\text{Side opposite } \angle 60^\circ}{\text{Side adjoining } \angle 60^\circ} = \frac{AR}{AB} = \frac{3.46}{2} = 1.73$$

$$\frac{\text{Side opposite } \angle 60^\circ}{\text{Side adjoining } \angle 60^\circ} = \frac{AR}{AB} = \frac{5.2}{3} = 1.73$$

$$\frac{\text{Side opposite } \angle 60^\circ}{\text{Side adjoining } \angle 60^\circ} = \frac{AR}{AB} = \frac{6.9}{4} = 1.72$$

Have the pupils consider only the first decimal place and see what conclusion they can make from the results. They will readily observe that all the quotients are the same. When asked what other fact is the same in each of the four, they will respond that it is the acute angle. It should now be very carefully pointed out that in each case the sides opposite and the sides adjoining the acute angles are of different lengths. Use this same procedure for each of the acute angles assigned.

The class is now ready to make the conclusion that in each group the result of dividing the side opposite the acute angle by the side adjoining the acute angle is the same for all angles of 30°, also for all

of 45°, and for all of 60°, and for all of 64°. Explain to the group that this result is in each case the tangent of the respective angle and that the relationship of these lines for each acute angle has been arranged for them in a tangent table.

This procedure will give the pupil a discover's feeling of ownership of the tangent table. He will realize that he has had a part in its construction and it will cease to be just a page to aid him in getting the answer to some problem. Many pupils as a result of this experiment, will want to make tangent tables of their own to compare with the one in their text. If they construct three triangles and take the average, their result will be faily accurate. This makes a good class project and it gives the pupil a real understanding of what a tangent is.

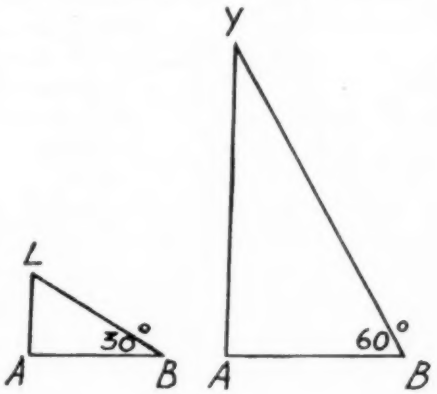
The formal definition is no longer "formal" or, as has often been the case, "formidable." The fact that in a right triangle the tangent is the ratio of the side opposite the acute angle to the side adjoining the acute angle, has been learned through discovery and not through memorization.

Now have the class consider two triangles. Take one from the 60° group and the other from the 30°. Have the pupils in the angle 60° group consider the angle at *Y*. Have them determine its size and discover its tangent ratio. Have those in the 30 angle group consider in the same manner the angel at *L*. This will emphasize that all right triangles have two acute angles and serve to show further that the tangent ratio is always the same for angles of the same size regardless of their position in the right triangle.

The pupils are now ready to go out doors with a transit and measure heights of objects. That will make the meaning of the angle of elevation clear and will also teach them the reason for considering the height of the instrument.

The angle of depression can be taught from a classroom window. A second or third floor window is desirable for this project. They can find the distance from the base of the building to various trees and objects. They will also enjoy having an imaginary lake beside the building and measuring its width. They will be eager to make these measurements with a tape to see how accurate they were.

The pupils will have so much enthusiasm for this work that many of them will



want to make transits of their own so that they can make measurements in their leisure time in their own neighborhoods. In our school we have had several made and to date no two have been alike in all respects. Each has however measured satisfactorily horizontal and vertical angles. It has also given its creator keen satisfaction and enjoyment.

This same method can be used with sines and cosines to enable the pupils to discover for themselves what those ratios mean.

When the pupil has had an opportunity to make real applications and to learn through this laboratory method, the principles have more meaning and are more readily retained in his memory. This active learning will have given the pupils an experience which they can not readily forget. It will have taken the tangent from the textbook and given it life and vitality. Each pupil will have become a discoverer in his own right and will have had a learning thrill that is a part of laboratory learning.

EDITORIALS

How to Get New Members

As STATED in the March issue, one of the most important steps for The National Council of Teachers of Mathematics to take now is to start a drive for new members. In the first place the Council needs more members to carry on its work successfully. Secondly, teachers of mathematics all over the country need the help and inspiration that permanent membership in the Council can give. Right now there is a shortage of teachers in this country and school authorities everywhere are beginning to see that salaries of mathematics teachers must be raised unless large numbers of children are to be left with no teachers at all or poor ones at best. Other things being equal, the larger the membership of the Council the more power it will have to demand and secure better salaries, so that abler men and women will be attracted to teaching. Of course salary alone has not and should not be the deciding factor in choosing teaching as a life profession, but salaries must be raised. In union there is strength. George Washington is reported to have said, "A hundred thousand men coming one after another could not move a ton weight, but the united strength of fifty would transport it with ease." Just so!

In line with our campaign to increase membership many local groups are accepting responsibility for collecting $2 in dues for The National Council as well as their local dues for each year. The ideal goal for any group which so affiliates with the Council is 100%. The work in each state should not be left entirely to the State Representatives although in most states they are doing excellent work. No meeting should be held and no program given without some member of the Council giving a good presentation, if only for a few minutes, of the work the Council is trying to do. In such a talk reference, of course, should be made to THE MATHEMATICS TEACHER (the official journal of the Council) and to The National Council Yearbooks. If groups all over the country would carry on such a program continuously, as some of our most vigorous groups do, we would get more members, would be much more powerful, and could carry on our work better for the good of all concerned.—W.D.R.

Kenneth Brown Resumes Duties as Chairman of State Representatives

DR. KENNETH BROWN, formerly Chairman of the State Representatives of The National Council of Teachers of Mathematics, and recently a Lieutenant Commander in the Navy, has resumed his duties for the Council and will carry on the good work that he started so well before he joined the Navy. He is now head of the department of mathematics at The Eastern State Teachers College at Greenville, North Carolina, THE MATHEMATICS TEACHER wishes Dr. Brown the greatest success.—W.D.R.

Members Whose Subscriptions Expire in May

MEMBERS whose subscriptions expire in May 1946 should renew their memberships at once so as not to miss the October issue. To be sure, membership in the Council does not expire until October, but there are no issues of the magazine in June, July, August and September. Members often act under the impression that they do not need to renew until October. This causes us a great deal of unnecessary trouble and delay. We hope that subscribers will renew promptly.—W.D.R.

◆ IN OTHER PERIODICALS ◆

By NATHAN LAZAR

Midwood High School, Brooklyn 10, New York

American Mathematical Monthly

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1. "An Old Letter," p. 1.
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3. Alaoglu, Leonidas, and Giese, J. H., "Uniform Isohedral Tori," pp. 14–20.
4. Everett, C. J., and Ryser, H. J., "The Gram Matrix and Hadamard Theorem." pp. 21–23.
5. Greenwood, R. E., and Gleason, A. M., "Binomial Identities," pp. 24–27.
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2. MacDuffee, C. C., "Education Fitted to the Capacity of the Student," pp. 21–25.
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4. Ruth, Mary A., "The Meaning of Words in Mathematics," pp. 28–32.
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2. "Technical Mathematics: A Discussion," pp. 145–160.
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3. "Syllabuses for Examinations taken by Sixth-Form Pupils: "A Discussion," pp. 161–177. Snell, K. S., Maxwell, E. A., Bereton, L. J., Durell, C. V., Chamberlain, K. S., Wolstenholme, E. O., Busbridge, I. W., Wardle, K. L., Smith, J. W. Ashley, Bearney, K. J., Williams, E. M., Barton, A., Powell, F. C., Hope-Jones, W., McIntosh, K. W., Sibson, R., Pickford, P. M., Morris, T. D.
4. Mathematical Note.
   1842. Lewis, E. P., "On Note 1727: Extension of Simson's Line."
5. "Gleanings Far and Near" (1462–1465).

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    1846. Wright, C. E., "Solutions of Linear Differential Equations with Constant Coefficients."
    1847. Lyness, R. C., "Cycles."
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2. "The Preparation of High School Science and Mathematics Teachers: Report No. 4 of the AAAS Cooperative Committee on Science Teaching," pp. 107–118.
3. Carnahan, Walter H., "History of Algebra" (II), pp. 125–130.
4. Ransom, William R., "A Geometric Derivation of Formulas for Tan $(A+B)$ and Tan $(A-B)$," pp. 139–140.
5. Niessen, A. M., "Marking on a Curve," pp. 155–158.
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5. "What is Problem Solving in Arithmetic?" pp. 256–266. Discussion led by J. T. Johnson.
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   (b) Buswell, G. T., "Is It Number Relations?"
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7. Nyberg, Joseph A., "Notes from a Mathematics Classroom" (continued, 112–115), pp. 277–280.
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BOOK REVIEW

*Productive Thinking.** By Max Wertheimer. New York: Harper & Brothers, 1945. Pp. 224.

The late Max Wertheimer (1880–1943) worked on this little book for seven years, hampered by his language difficulties, the inadequacy of existing psychological terms to express his ideas properly, and the goal of perfectionism which he held high for himself and his students. The result, well worth the effort exerted, is an informal, often dramatic description of what takes place during productive thinking, offered to the scientific psychologist, the logician, and the educator "primarily as an invitation to reconsider basic issues" (4).

Wertheimer was very much concerned with problems of teaching and learning. "Although there are good teachers with a natural feeling for what genuine thinking means, the situation in schools is often not good. How teachers act, how a subject matter is taught, how textbooks are written, all this is widely determined by two traditional views about the nature of thinking: the view of traditional logic and the view of association theory" (2–3). The Introduction contains a brief analysis of these two theories, neither of which, it is claimed, is adequate to characterize truly productive thinking; they do not distinguish between good and poor thinking processes.

Various examples of thinking, ranging from simple, modest ones to some of the finest, are discussed. The nature of the illustrations is revealed by the chapter titles. (1) The area of the parallelogram, (2) The problem of the vertical angles, (3) The famous story of young Gauss, (4) Two boys play badminton; a girl describes her office, (5) Finding the sum of the angles of a polygon, (6) A discovery by Galileo, and (7) Einstein: the thinking that led to the theory of relativity. One notices a predominance of mathematical examples, chosen because "they are tasks in which it is possible to proceed in a transparent, crystal-clear way . . . (and) it seemed better first to discuss the basic theoretical issues in structurally simpler materials" (78).

Chapter 1 deals with methods of teaching the concept of area; on the one hand, methods which lead merely to mechanical repetition of what was taught by the teacher, without much understanding, and on the other, methods which lead to insight into the structure of the problem and which make for sensible transfer to other, superficially different situations. It is here that one finds precise illustrations of such Gestalt concepts as characteristic form, structurization, whole process, reorganization and recentering. Some of the thought processes encountered when subjects were given the problem of proving two vertical angles equal, are then studied. Their main difficulty was the failure to realize the need for proof. A refreshing approach to finding the sum of certain regular series is presented in the subsequent chapter; an appendix is devoted to a description of the author's train of thought in attempting to understand the structural relation between an infinite geometric series with a ratio less than one, and the formula for the sum.

Wertheimer was shocked at the mechanical, non-insightful manner in which some individuals approached mathematical problems. They attempted to subsume them under a set pattern of familiar problems, applied long, tedious procedures where more direct ones would serve, and even when they applied a formula correctly, did not understand the inner connection between the formula and the problem. Thus, when children were given tasks similar to these:

$$\frac{271+272+273+274+275}{5}=?\text{ and}$$

$$\frac{274+274+274+274+274}{5}=?$$

a number of them "who were especially good in arithmetic in their school were entirely blind, started at once with tedious figuring or begged to be excused from the cumbersome task" (112). Such "foolish reactions," it is maintained, are not characteristic of children's behavior, but are fostered by certain school procedures: the emphasis on drill, on mechanical repetition, on instantaneous responses, and on the development of piecemeal habits (112).

To illustrate the concepts of reorganization and recentering, two examples from everyday life are dealt with in Chapter 4. (1) Two boys are playing badminton; suddenly, the younger boy, the poorer player, refuses to continue. The older, anxious to play, is faced with a problem which he solves productively by changing the character of the game from a competitive one to a cooperative one. (2) A young girl, a clerk in an office, describes her place of work, relating all things to herself, the ego. The result is bewildering, giving the listener the impression that she is the boss. "Real thinkers forget about themselves in thinking. The main vectors in genuine thought often do not refer to an I with its personal interests" (135). One cannot help wondering about the effect on genuine thinking

* The manuscript, completed shortly before Wertheimer's death, was prepared for publication by S. E. Asch, W. Kohler, and C. W. Meyer, who made "certain linguistic revisions" (xi).

had by the competitive, ego-centered atmosphere prevailing in and fostered by our educational system.

In the fifth chapter, Wertheimer describes his discovery of the functional interrelatedness between closed figures and the sum of their interior or exterior angles, relations which he found carried over from two-dimensional to three-dimensional figures. This discovery provided him with many joyful moments, and should do the same for the reader. There follows a dramatic presentation of the possible reasoning used by Galileo in deriving the law of inertia. The seventh chapter is devoted in the main to the role played by the concept of "simultaneity" in the development of the theory of relativity, based on information obtained in long discussions with his intimate friend, Albert Einstein. A conclusion entitled "Dynamics and logic of productive thinking," provides a summary as well as a theoretical discussion of traditional approaches and of the Gestalt interpretation of thinking.

When the reviewer first met Max Wertheimer, in 1938, this psychologist, philosopher, and expert musician told her, "At times I find mathematics my only consolation." Teachers of mathematics may find in this book inspiration for better, more productive teaching, directed toward helping others to find "Consolation in the beauties of this subject."

EDITH H. LUCHINS

◆ NEWS NOTES ◆

The Association of Teachers of Mathematics in New England held a dinner meeting at Boston on January 26, 1946 at the Y.W.C.A. Professor Marshall H. Stone of Harvard University was the guest speaker. His topic was "Innovations in Mathematics Instruction."

Officers 1946

Prof. Titus E. Mergendahl, Pres., Tufts College, Medford, Mass.
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Dr. Nathan Lazar of Midwood High School, Brooklyn, N. Y., was the dinner speaker at the last meeting of Section 11 (formerly Section 19) of the New York Society For The Experimental Study of Education at the Men's Faculty Club of Columbia University on Friday, March 1, 1946. His topic was "The Invalidity of the Analysis of The Indirect Proof in Geometry Textbooks." Mr. Joseph Orleans, of George Washington High School, New York City, was the speaker at a previous meeting on Friday, Feb. 8, 1946. His topic was, "The Tentative Six-Year Plan in Mathematics Proposed by the New York State Department of Education."

The Rhode Island Mathematics Teachers' Association held its annual spring meeting at Brown University in Providence on Saturday, March 16, 1946. Dr. Julian Lowell Coolidge spoke on the topic "Euler's Polyhedra." Mr. M. L. Herman is president and Miss Harriet Whitaker is secretary-treasurer.

The Oklahoma branch of The National Council of Teachers of Mathematics held a luncheon meeting at Oklahoma City on Friday, Feb. 15, 1946 in connection with a meeting of The Oklahoma Education Association. The chairman was Miss Rosalie Long of Okmulgee and the secretary was Miss Kathleen Bagley of Tulsa. There were 200 teachers present. The program was as follows:

Woodwind Trio—Classen High School, Oakley Pittman, Director
"The Improvement of High School Mathematics Courses as Recommended by the Commission on Post-War Plans"—James H. Zant, Oklahoma, A. & M.
Northeast High School A Cappella Choir, Elna Smith, Director
"Applications of Mathematics in the Field of Artillery"—Robert S. Stafford, Gunnery Department, Fort Sill
"Improving the Child's Understandings and Skills in Arithmetic"—W. C. Jones, Dean of Graduate School, George Peabody College for Teachers, Nashville, Tennessee

Officers elected for the ensuing year are:

President.—Mrs. Katherine Levert, Oklahoma City
Vice-President.—Miss Rosalie Long, Okmulgee
Secretary-treasurer.—Miss Kathleen Bagley, Tulsa.

Miss Eunice Lewis, State Representative of The National Council of Teachers of Mathematics, reported the meeting.

A Joint Dinner Meeting of the Men's Mathematics Club and the Women's Mathematics Club of the Chicago and Metropolitan Area was held at the Central Y.M.C.A. on Friday, March 15, 1946. Dr. Glenn T. Seaborg, Professor, Department and College of Chemistry of the University of California at Berkeley, was the guest speaker. His topic was "Atomic Power."

Dr. Seaborg's main field of investigation has been nuclear chemistry and nuclear physics. He is co-discoverer of plutonium and elements 95 and 96. On leave of absence from the University of California, he has been working on the application of plutonium to atomic energy at the Metallurgical Laboratory of the University of Chicago for the last three and one half years. He was primarily responsible for the development of the chemical separation procedures which were used in the manufacture of plutonium at Clinton, Tennessee, and at Hanford, Washington.

Since 1936, Dr. Seaborg has published more than 40 papers on the general subject of nuclear physics, artificial radioactivity and the applications of artificial radioactivity to chemistry.

Glenn Anderberg, Secretary-Treasurer — Edward Tierney, Program Chairman

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The Rocky Mountain Section of The National Council of Teachers will join the Rocky Mountain Section of The Mathematical Association of America and the Mathematics Section, Eastern Division of the Colorado Education Association in a joint meeting to be held April 19th and 20th at Boulder, Colorado. The theme of the meeting is to be "Mathematics in a Changing World." Mr. William Betz, Specialist in Mathematics in the Schools of Rochester, New York, and a member of the Commission on Post War Plans will deliver several addresses as will some of our local members including Dr. J. R. Britton of the University of Colorado and President of the Rocky Mountain Section of The Mathematical Association of America.

FRIDAY, APRIL 19, 1946

Presiding: DR. J. R. BRITTON

2:30 P.M. Several short mathematical papers on the College level.
Address: The present educational situation and the resulting crisis in mathematics—William Betz, Specialist in Mathematics in the Schools of Rochester, New York.

6:30 P.M. Banquet. Blanchard's Lodge.
Greetings: President Stearns, University of Colorado.
Response: Professor C. F. Barr, University of Wyoming.
Address: The Laplace transformation—Dr. J. R. Britton, University of Colorado.

SATURDAY, APRIL 20, 1946

9:00 A.M. Business session of The Rocky Mountain Section of The Mathematical Association of America.

9:30 A.M. Section meetings.

A. Elementary School-Junior High School Section.
Presiding: Professor A. E. Mallory, Colorado State College of Education.
Discussion: Modern trends in teaching the fundamentals of Mathematics.
Leaders: Mrs. Esther Taylor, Denver Public Schools, Grades 1–3. Miss Josephine Bellodi, Denver Public Schools, Grades 4–6. Leader for Grades 7–9 to be announced later.

B. Senior High School-College Section.
Presiding: Harry W. Charlesworth, East High School, Denver.
Discussion: What can we learn about needed changes in the content of our mathematics courses or in methods of instruction as indicated by the experiences of the men in the armed services?
Leaders: Russell L. Casement, Denver Public Schools, Fred J. Clark, Colorado A. & M. College, Carl Swanson, University of Denver, Mr. Klooster, University of Colorado.

10:30 A.M. General Session.
Presiding: Kenneth E. Gorsline, East High School, Denver.
Address: The necessary reconstruction of Mathematics in the light of war experience—William Betz, Specialist in Mathematics in the Schools of Rochester, New York.

11:30 A.M. Forum Discussion: "The work of the Commission on Post War Plans." Leader: William Betz.

1:30 P.M. Luncheon. Wayne's Cafe.

Reservations for the banquet and luncheon should be made immediately with Dr. A. J. Lewis, Pearl 0081, 2136 South Josephine Street, Denver, Colorado. Hotel reservations should be made directly with the hotels. The Boulderado has single rooms at $1.50 and up and double rooms at $2.50 and up. The Albany has double rooms at $2.50 and up.

An Institute for Teachers of Mathematics (Arithmetic-into-Calculus) will be held at Duke University, August 8–17. Professor W. W. Rankin will be the director and Miss Veryl Schult the assistant director. The complete program will appear in the May issue of THE MATHEMATICS TEACHER.

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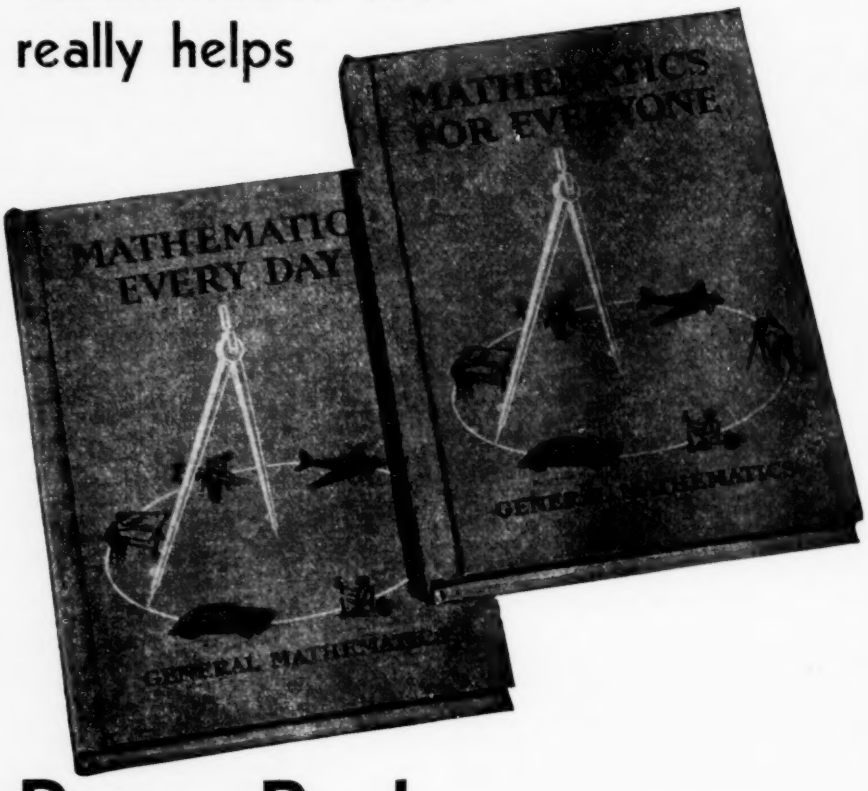

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